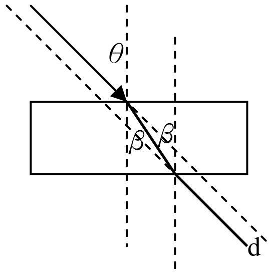
教育部 100 學年度高級中學數理及資訊學科能力競賽物理科決賽

筆試試題（一）參考解

【第一題參考解】

(a)



上邊界

$\sin\theta_{in} = n\sin\beta$

下邊界

$n\sin\beta = \sin\theta_{out} = \sin\theta_{in}$

設光線在平板中的路徑長為 ℓ，則 $\ell = \dfrac{t}{\cos\beta}$

$$d = \ell\sin(\theta-\beta) = \ell(\sin\theta\cos\beta - \cos\theta\sin\beta)$$

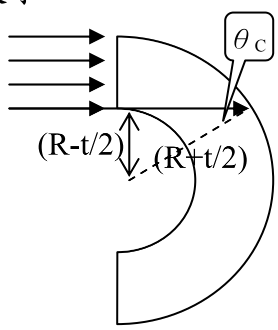
$$d = \frac{t}{\cos\beta}(\sin\theta\cos\beta - \cos\theta\sin\beta) = t(\sin\theta - \frac{\cos\theta\sin\beta}{\cos\beta})$$

$$\sin\theta = n\sin\beta \to \sin\beta = \sin\theta/n, \quad \cos\beta = \sqrt{1-\sin^2\theta/n^2}$$

$$d = t(\sin\theta - \frac{\cos\theta\sin\theta/n}{\cos\beta}) = t\sin\theta(1-\frac{\cos\theta}{n\sqrt{1-\sin^2\theta/n^2}})$$

(b)

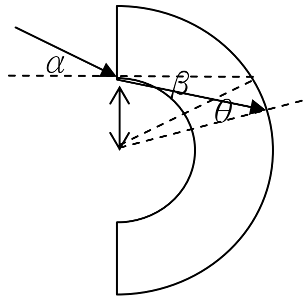
下圖四光線中，上方入射角最大，下方入射角最小。



$$\sin\theta_C = \frac{1}{n} = \frac{R_C - t/2}{R_C + t/2}$$

$$R_C = \left(\frac{n+1}{n-1}\right)\frac{t}{2}$$

(c)



$$\sin\alpha = n\sin\beta,\quad \sin\theta_C = \frac{1}{n}$$

Bottom Ray:

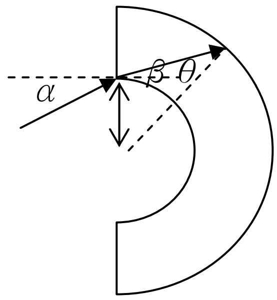
$$\frac{\sin\theta}{R-t/2} = \frac{\sin(\pi/2-\beta)}{R+t/2} = \frac{\cos(\beta)}{R+t/2}$$

$$\sin\theta = \frac{(R-t/2)\cos(\beta)}{R+t/2} \ge \frac{1}{n}$$

$$\cos(\beta) = \sqrt{1-\sin^2\beta} \ge \frac{R+t/2}{n(R-t/2)}$$

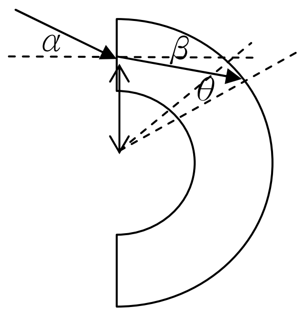
$$\sqrt{n^2-n^2\sin^2\beta} = \sqrt{n^2-\sin^2\alpha} \ge \frac{R+t/2}{(R-t/2)} \to n^2-\sin^2\alpha \ge \left(\frac{R+t/2}{R-t/2}\right)^2$$

$$n^2-\left(\frac{R+t/2}{R-t/2}\right)^2 \ge \sin^2\alpha \to \sin\alpha \le \sqrt{n^2-\left(\frac{R+t/2}{R-t/2}\right)^2}$$



$$\frac{\sin\theta}{R-t/2} = \frac{\sin(\beta+\pi/2)}{R+t/2} = \frac{\cos(\beta)}{R+t/2}$$ ，同前

Middle Ray:



$$\frac{\sin\theta}{R} = \frac{\sin(\pi/2-\beta)}{R+t/2} = \frac{\cos(\beta)}{R+t/2}$$

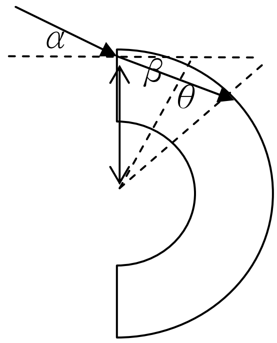
$$\sin\theta = \frac{(R)\cos(\beta)}{R+t/2} \ge \frac{1}{n}$$

$$\cos(\beta) = \sqrt{1-\sin^2\beta} \ge \frac{R+t/2}{nR}$$

$$\sqrt{n^2-n^2\sin^2\beta} = \sqrt{n^2-\sin^2\alpha} \ge \frac{R+t/2}{R} \to n^2-\sin^2\alpha \ge \left(\frac{R+t/2}{R}\right)^2$$

$$n^2-\left(\frac{R+t/2}{R-t/2}\right)^2 \ge \sin^2\alpha \to \sin\alpha \le \sqrt{n^2-\left(\frac{R+t/2}{R}\right)^2}$$

Top Ray:



$$\frac{\sin\theta}{R+t/2}=\frac{\sin(\pi/2-\beta)}{R+t/2}=\frac{\cos(\beta)}{R+t/2}$$

$$\sin\theta=\cos(\beta)\geq\frac{1}{n}\quad\cos(\beta)=\sqrt{1-\sin^2\beta}\geq\frac{1}{n}$$

$$\sqrt{n^2-n^2\sin^2\beta}=\sqrt{n^2-\sin^2\alpha}\geq 1\rightarrow n^2-\sin^2\alpha\geq 1$$

$$n^2-1\geq\sin^2\alpha\rightarrow\sin\alpha\leq\sqrt{n^2-1}$$ Plane Wave guide

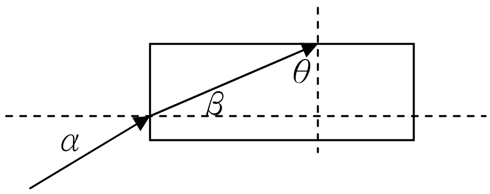
The bottom ray limits the maximum incident angle

$$\alpha\leq\sin^{-1}\sqrt{n^2-\left(\frac{R+t/2}{R-t/2}\right)^2}$$

(d)

未鍍膜時

$$\sin\alpha=n\sin\beta,\quad n\sin\theta_C=\sin\frac{\pi}{2}=1$$



鍍膜後

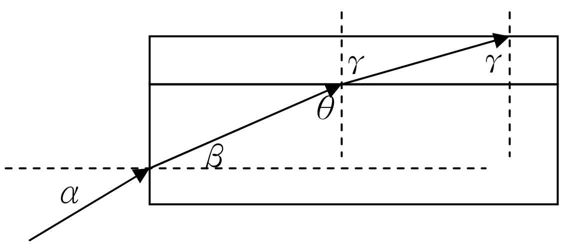
$$\sin\alpha=n\sin\beta$$

$$n\sin\theta=n_B\sin\gamma$$

$$n_B\sin\gamma_C=\sin\frac{\pi}{2}=1$$

$$n\sin\theta_C=n_B\sin\gamma_C=1$$

與未鍍膜相同



【第二題參考解】

由題目可知

熱導率 K = 0.055 W/m-K

內牆之熱傳導係數 h_i = 8.5 W/m^2-K

外牆之熱傳導係數 h_o = 34 W/m^2-K

室內溫度 T_i = 20 ℃ = 293 K

室外溫度 T_o = -20 ℃ = 253 K

牆壁面積 A = 200 m^2

牆壁厚度 D = 15 cm = 0.15 m

假設此屋處於熱傳導穩定狀態(steady state)，並定義

1. 內牆與室內空氣之熱流率 $\dot{q}_i = \frac{\delta q_i}{dt} = \frac{1}{A}\frac{\delta Q}{dt}$, where Q = heat
2. 牆內部之熱流率 $\dot{q}_w$
3. 外牆與戶外空氣之熱流率 $\dot{q}_o$

$\Rightarrow \dot{q}_i = \dot{q}_w = \dot{q}_o = \dot{q}$

(1) 計算屋內熱流率： $\dot{q}_i = h_i \times A \times (T_i - T_{wi})$

(2) 計算屋外熱流率：$\dot{q}_o = h_o \times A \times (T_{wo} - T_o)$

(3) 計算牆內熱流率：$\dot{q}_w = -K \times A \times \frac{dT}{dx} \approx -K \times A \times \frac{T_{wo} - T_{wi}}{D}$

(4) 由(1)式可得：$T_{wi} = T_i - \frac{\dot{q}}{h_i \times A}$

(5) 由(2)式可得：$T_{wo} = T_o + \frac{\dot{q}}{h_o \times A}$

(6) 將(4),(5)兩式代入(3)式中：

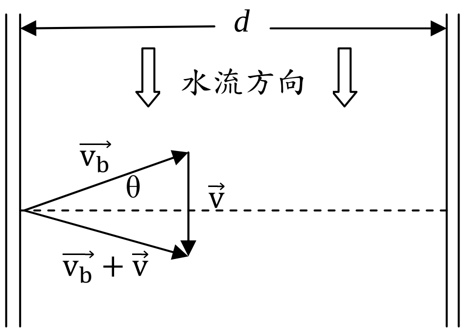
$$\dot{q} = K \times A \times \frac{T_{wi} - T_{wo}}{D} = \frac{K \times A}{D}\left[(T_i - T_o) - \frac{\dot{q}}{A} \times (\frac{1}{h_i} + \frac{1}{h_o})\right]$$

$$\Rightarrow \dot{q} = \frac{A \times (T_i - T_o)}{(\frac{1}{h_i} + \frac{D}{K} + \frac{1}{h_o})} \cong 2.78 \text{（kW）}$$

(7) 假設恆溫系統效率為 1。因此，每月至少需付出電費=2.78x24x30x3=6,000 元。

【第三題參考解】

假設船相對於河水速度$\overrightarrow{v_b}$，水流速度為$\vec{v}$，則船相對於岸上的速度為$\overrightarrow{v_b}+\vec{v}$。



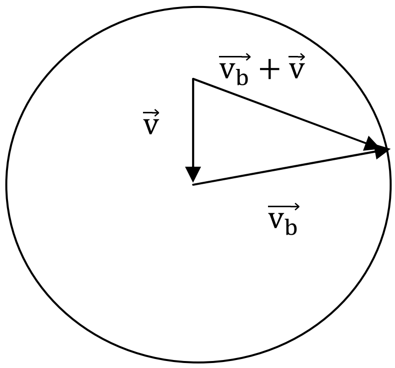
如上圖，則渡河時間為

$$t=\frac{d}{v_b\cos\theta}$$

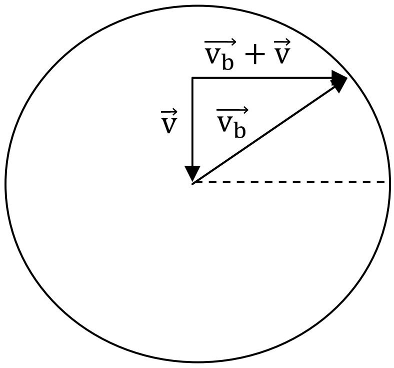
第一次渡河要求時間最短，則$\theta=0,\ \cos\theta=1$，所以$T = d/v_b$。

第二次渡河

(a) $\mathbf{v_b} > \boldsymbol{v}$ **三種速度的向量圖改為如下**，其中圓是以 v_b 為半徑所繪，

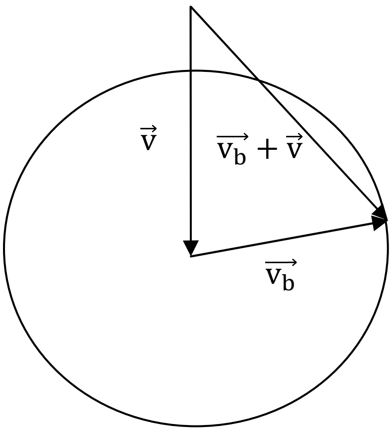


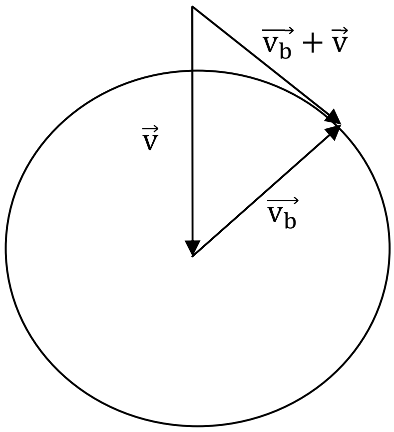
$\overrightarrow{v_b}+\vec{v}$將落在圓上任一點。因為要求偏移距離最小，代表$\overrightarrow{v_b}+\vec{v}$在水流方向的分量最小，即$\overrightarrow{v_b}+\vec{v}$與$\vec{v}$夾角最大(不超過直角)，如下圖



所以$2T = d/\sqrt{v_b^2-v^2}$。已知$T = d/v_b$，所以$v=\frac{\sqrt{3}}{2}\frac{d}{T}$。

(b) $\mathbf{v_b} < v$ 三種速度的向量圖改為如下左圖，





因為要求偏移距離最小，代表$\overrightarrow{v_b}+\vec{v}$在水流方向的分量最小，$\overrightarrow{v_b}+\vec{v}$與$\vec{v}$夾角最大如上右圖，即$\overrightarrow{v_b}+\vec{v}$切圓於一點，所以$2T = d/(\sqrt{v^2 - v_b^2}\frac{v_b}{v})$ ，已知$T = d/v_b$，所以$v = \frac{2}{\sqrt{3}}\frac{d}{T}$。

【第四題參考解】

假設初始態的溫度為 T_1,壓力為 P_1,體積為 V_1

∴所以滿足三方程式　(1)　$P_1V_1=nRT_1$

(2)　$P_1S=kh_1$

(3)　$V_1=h_1S$

$$\therefore\ P_1V_1=\frac{kh_1}{S}\cdot h_1S=kh_1^2=nRT_1\quad \therefore T_1=\frac{kh_1^2}{nR}$$

同理 當吸收 Q 熱量時 活塞高度為 h_2 則 $kh_2^2=nRT_2\quad \therefore T_2=\frac{kh_2^2}{nR}$

由熱力學第一定理: $Q=\Delta U+\Delta W$

因為 $\Delta W=\frac{1}{2}k(h_2^2-h_1^2)$ 而 $\Delta U=nC_V(T_2-T_1)=\frac{C_V}{R}k(h_2^2-h_1^2)$

$$\therefore\quad h_2=\sqrt{h_1^2+\frac{2QR}{k(R+2C_V)}}$$

$$\Delta T=T_2-T_1=\frac{2Q}{n(2C_V+R)}$$

【第五題參考解】

(a) No, $\because \vec{E}_1 \bullet \vec{E}_2^* = \vec{E}_1^* \bullet \vec{E}_2 = 0$,

$$I = \left|\vec{E}\right|^2 = (\vec{E}_1 + \vec{E}_2) \bullet (\vec{E}_1^* + \vec{E}_2^*) = (E_{1x}\hat{x} + E_{2y}\hat{y}) \bullet (E_{1x}^*\hat{x} + E_{2y}^*\hat{y}) = \left|E_{1x}\right|^2 + \left|E_{2y}\right|^2$$

(b) $I = \left|\vec{E}\right|^2 = (\vec{E}_1 + \vec{E}_2) \bullet (\vec{E}_1^* + \vec{E}_2^*) = (E_{1x}\hat{x} + E_{2x}\hat{x}) \bullet (E_{1x}^*\hat{x} + E_{2x}^*\hat{x})$

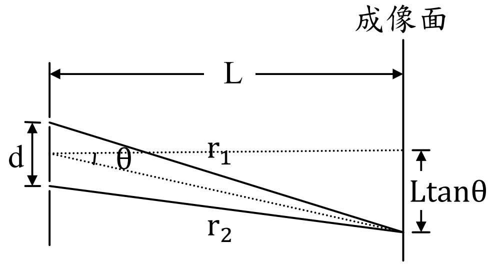
$$= \left|E_{1x}\right|^2 + \left|E_{2x}\right|^2 + E_{1x}E_{2x}^* + E_{1x}^*E_{2x}$$

$$= \left|E_{1x}\right|^2 + \left|E_{2x}\right|^2 + \left|E_{1x}\right|e^{i\phi_1}\left|E_{2x}\right|e^{-i\phi_2} + \left|E_{1x}\right|e^{-i\phi_1}\left|E_{2x}\right|e^{i\phi_2}$$

$$= E_{1x0}^2 + E_{2x0}^2 + E_{1x0}E_{2x0}e^{i(\phi_1-\phi_2)} + E_{1x0}E_{2x0}e^{-i(\phi_1-\phi_2)}$$

$$= E_{1x0}^2 + E_{2x0}^2 + 2E_{1x0}E_{2x0}\cos(\phi_1 - \phi_2)$$

$$\phi_1 - \phi_2 = \frac{2\pi r_1}{\lambda} - \frac{2\pi r_2}{\lambda}$$



建設性干涉條件：$\phi_1 - \phi_2 = 2\pi k$

$\Rightarrow r_1 - r_2 = k\lambda$

$\because r_1 - r_2 \cong d\sin\theta$

$\therefore d\sin\theta = k\lambda, \quad k = 0, \pm1, \pm2\cdots$

$$bright\ \ fringes\ \ from\ center = L\tan\theta \approx L\theta \approx L\sin\theta = \frac{Lk\lambda}{d}, \quad k = 0, \pm1, \pm2\cdots$$

$$\therefore period = \frac{L\lambda}{d}$$

(c) OPD = $n\ell - \ell = (1.5 - 1)\,\ell = 0.5\ell = (k + \frac{1}{2})\lambda_0 \quad \Rightarrow \quad \ell = (2k+1)\lambda_0, \ k = 0,1,2,\cdots$

Or

$$I = \left|\vec{E}\right|^2 = \left|\vec{E}_1 + \vec{E}_2\right|^2 = \left|E_{1x}\right|^2 + \left|E_{2x}\right|^2 + 2E_{1x0}E_{2x0}\cos(\phi_1 - \phi_2)$$

$$\phi_1 - \phi_2 = \frac{2\pi n\ell}{\lambda_0} - \frac{2\pi\ell}{\lambda_0} = (2k+1)\pi \Rightarrow \ell = (2k+1)\lambda_0, \quad k = 0,1,2,\cdots$$

(d) $I_{\max} = E_{1x0}^2 + E_{2x0}^2 + 2E_{1x0}E_{2x0}$

$$I_{\min} = E_{1x0}^2 + E_{2x0}^2 - 2E_{1x0}E_{2x0}$$

$$V = \frac{2E_{1x0}E_{2x0}}{E_{1x0}^2 + E_{2x0}^2} = \frac{1}{5}$$

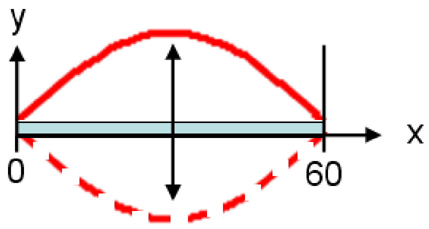
$$\therefore E_{1x0}^2 - 10E_{1x0}E_{2x0} + E_{2x0}^2 = 0$$

$$\Rightarrow \frac{E_{1x0}}{E_{2x0}} = 5 \pm \sqrt{24} \quad (take + sign)$$

$$\because \frac{I_1}{I_2} = \frac{A_1}{A_2} = \frac{D_1^2}{D_2^2} \Rightarrow \frac{E_{1x0}}{E_{2x0}} = \frac{D_1}{D_2}$$

【第六題參考解】

(a) 由實驗觀察基頻的運動模式（駐波）如下示意圖，



可以假設波函數為

$$y(x,t)=(C\cos kx+D\sin kx)(E\cos\omega t+F\sin\omega t),\qquad \omega=kv$$

要求此解滿足邊界條件 $y(x=0,t)=0, y(x=L,t)=0$

可得

$$C(E\cos\omega t+F\sin\omega t)=0\Rightarrow C=0$$

$$(D\sin kL)(E\cos\omega t+F\sin\omega t)=0\Rightarrow \sin kL=0\Rightarrow kL=n\pi\Rightarrow k=\frac{n\pi}{L},\qquad n=1,2,3,...$$

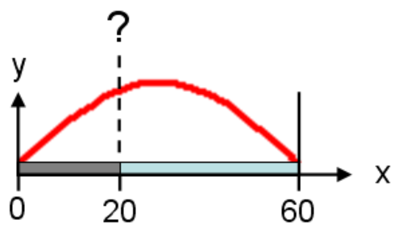
$$\therefore y(x,t)=\sin\frac{n\pi x}{L}(E\cos\omega t+F\sin\omega t)=A\sin\frac{n\pi x}{L}\cos(\omega t+\phi)$$

基頻的波函數就是 n=1 的 y(x,t)，

$$y(x,t)=\sin\frac{\pi x}{L}(E\cos\omega t+F\sin\omega t)=A\sin\frac{\pi x}{L}\cos(\omega t+\phi),\omega=kv=\frac{\pi}{L}\sqrt{\frac{F_t}{\mu}}$$

其中 A,ϕ 由初始條件決定。

(b) 對於兩段不同線密度的弦來說，弦上每個位置的振盪頻率相同。



參考解答(a)之駐波函數，將左右兩段弦之波函數寫成

$$y_{左}(x,t)=\sin k_{左}x(E_{左}\cos\omega t+F_{左}\sin\omega t),$$

$$y_{右}(x,t)=\sin k_{右}(L-x)(E_{右}\cos\omega t+F_{右}\sin\omega t),$$

由於振盪頻率相同，所以

$$\omega=k_{左}v_{左}=k_{右}v_{右}\Rightarrow k_{左}\sqrt{\frac{F_t}{4\mu}}=k_{右}\sqrt{\frac{F_t}{\mu}}\Rightarrow k_{左}=2k_{右}$$

所以波函數可以寫成

$$y_{左}(x,t)=\sin k_{左}x(E_{左}\cos\omega t+F_{左}\sin\omega t)$$

$$y_{右}(x,t)=\sin\frac{k_{左}}{2}(L-x)(E_{右}\cos\omega t+F_{右}\sin\omega t)$$

接著考慮左段弦與右段弦在交界處之波動函數連續性，在任意時刻滿足條件一

$y_{左}(x=\frac{L}{3},t)=y_{右}(x=\frac{L}{3},t)$ 以及條件二 $\left.\frac{\partial y_{左}(x,t)}{\partial x}\right|_{x=\frac{L}{3}}=\left.\frac{\partial y_{右}(x,t)}{\partial x}\right|_{x=\frac{L}{3}}$

依據條件一

$$\sin\frac{k_{左}L}{3}(E_{左}\cos\omega t+F_{左}\sin\omega t)=\sin\frac{k_{左}L}{3}(E_{右}\cos\omega t+F_{右}\sin\omega t)$$

$$(E_{左}\sin\frac{k_{左}L}{3}-E_{右}\sin\frac{k_{左}L}{3})\cos\omega t+(F_{左}\sin\frac{k_{左}L}{3}-F_{右}\sin\frac{k_{左}L}{3})\sin\omega t=0$$

$$\Rightarrow\begin{cases}E_{左}\sin\frac{k_{左}L}{3}-E_{右}\sin\frac{k_{左}L}{3}=0\\ F_{左}\sin\frac{k_{左}L}{3}-F_{右}\sin\frac{k_{左}L}{3}=0\end{cases}$$

依據條件二

$$\cos\frac{k_{左}L}{3}(E_{左}\cos\omega t+F_{左}\sin\omega t)=-\frac{1}{2}\cos\frac{k_{左}L}{3}(E_{右}\cos\omega t+F_{右}\sin\omega t)$$

$$(E_{左}\cos\frac{k_{左}L}{3}+\frac{1}{2}E_{右}\cos\frac{k_{左}L}{3})\cos\omega t+(F_{左}\cos\frac{k_{左}L}{3}+\frac{1}{2}F_{右}\cos\frac{k_{左}L}{3})\sin\omega t=0$$

$$\Rightarrow\begin{cases}E_{左}\cos\frac{k_{左}L}{3}+\frac{1}{2}E_{右}\cos\frac{k_{左}L}{3}=0\\ F_{左}\cos\frac{k_{左}L}{3}+\frac{1}{2}F_{右}\cos\frac{k_{左}L}{3}=0\end{cases}$$

整理上述條件限制後，可得

$$\begin{cases}E_{左}\sin\frac{k_{左}L}{3}-E_{右}\sin\frac{k_{左}L}{3}=0\\ E_{左}\cos\frac{k_{左}L}{3}+\frac{1}{2}E_{右}\cos\frac{k_{左}L}{3}=0\end{cases}$$

$$\begin{cases}F_{左}\sin\frac{k_{左}L}{3}-F_{右}\sin\frac{k_{左}L}{3}=0\\ F_{左}\cos\frac{k_{左}L}{3}+\frac{1}{2}F_{右}\cos\frac{k_{左}L}{3}=0\end{cases}$$

因為 $E_{左},E_{右},F_{左},F_{右}$ 不能同時為零，所以行列式

$$\begin{vmatrix} \sin\frac{k_{左}L}{3} & -\sin\frac{k_{左}L}{3} \\ \cos\frac{k_{左}L}{3} & \frac{1}{2}\cos\frac{k_{左}L}{3} \end{vmatrix} = 0 \Rightarrow \sin\frac{k_{左}L}{3}\cos\frac{k_{左}L}{3} = 0 \Rightarrow \sin\frac{2k_{左}L}{3} = 0 \Rightarrow \frac{2k_{左}L}{3} = n\pi$$

$$\Rightarrow k_{左} = \frac{3n\pi}{2L}, n = 1,2,3\ldots$$

在基頻的狀況下 n=1,再考慮條件一之限制，$E_{左} = E_{右} = E, F_{左} = F_{右} = F$

所以

$$y_{左}(x,t) = \sin\frac{3\pi x}{2L}(E\cos\omega t + F\sin\omega t), \qquad 0 \le x < \frac{L}{3}$$

$$y_{右}(x,t) = \sin\frac{3\pi(L-x)}{4L}(E\cos\omega t + F\sin\omega t), \qquad \frac{L}{3} \le x < L$$

至於 E,F ，還是由初始條件決定，示意圖如下。