



# On the synchronization of uncertain chaotic systems

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## Abstract

In this article, we investigate the synchronization of uncertain chaotic systems. Based upon the parameters identification technique and a simple but efficient control method, we control the response system to be the drive system with parameters unknown. The techniques are successfully applied to Lorenz system and Chen system. Furthermore, the effect of external noise is under our discussion.

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## 1. Introduction

In the past decades, the investigation of chaotic synchronization has attracted a lot of attention owing to various potential applications, such as secure communication [1,2], animal and neuron systems [3,4], and the study of laser dynamics [5]. A general feature of the chaotic system is its extreme sensitivity to initial conditions. In other words, slight errors occurring in initial states of two similar oscillators will lead to completely different trajectories after enough transient time. Therefore, how to control two chaos systems to be synchronized has become an important topic in nonlinear science.

Since the study made by Pecora and Carroll [6,7], many other papers about controlling chaos systems to be synchronized have been published [8–18]. Most methods in those papers are valid only when parameters of the system are known. However, the study of uncertain system (parameters unknown) is quite important because it is hard to receive complete information about parameters in applications. Particularly, several authors have reported adaptive estimation techniques to attain chaos synchronization when the model parameters are unknown [19–24]. With these techniques, parameters identification and synchronization of chaotic systems can be achieved simultaneously. Recently, Yan and Li provide a control method to synchronize two identical chaotic systems via the Lyapunov stability theory [25]. It is natural but important to extend the idea to the chaotic systems with parameters unknown because the method is relatively simple and efficient. In this paper, we investigate the parameters identification technique to synchronize uncertain chaotic systems based on the method addressed by Yan and Li. The techniques are successfully applied to Lorenz system and Chen system. Furthermore, the effect of external noise is under our discussion.

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The rest of this paper is organized as follows. Section 2 presents the synchronization of uncertain Lorenz systems. Section 3 presents the result of Chen systems. Section 4 discusses the effect of external noise. Conclusions are finally drawn in Section 5.

## 2. Lorenz system

First, let us consider the synchronization of uncertain Lorenz systems. Suppose two Lorenz systems are coupled unidirectionally. That is, the second system is driven by the first one but the behavior of the first system is not affected by the second one. The two systems are called the master and slave systems individually.

The master system:

$$\begin{cases} \dot{x}_1 = \sigma_1(y_1 - x_1), \\ \dot{y}_1 = r_1x_1 - x_1z_1 - y_1, \\ \dot{z}_1 = x_1y_1 - b_1z_1, \end{cases} \quad (1)$$

where the parameters  $\sigma_1$ ,  $r_1$ , and  $b_1$  are fixed and unknown.

The slave system:

$$\begin{cases} \dot{x}_2 = \sigma_2(y_2 - x_2), \\ \dot{y}_2 = r_2x_2 - x_1z_2 - y_2 + u(x_2 - x_1), \\ \dot{z}_2 = x_1y_2 - b_2z_2, \end{cases} \quad (2)$$

where  $\sigma_2$ ,  $r_2$ , and  $b_2$  are the estimated value of unknown parameters and  $u \in R$  is the coupling parameter. We define the parameter errors as  $\tilde{\sigma} = \sigma_2 - \sigma_1$ ,  $\tilde{r} = r_2 - r_1$ ,  $\tilde{b} = b_2 - b_1$ , and the errors of two systems as  $e_1 = x_2 - x_1$ ,  $e_2 = y_2 - y_1$ ,  $e_3 = z_2 - z_1$ . It is easy to show that synchronization in the Lorenz system is a result of stable error dynamics between the drive system and the response system. Subtracting Eq. (1) from Eq. (2), a set of equations which govern the error dynamics are then given by

$$\begin{cases} \dot{e}_1 = \sigma_2(e_2 - e_1) + \tilde{\sigma}(y_1 - x_1), \\ \dot{e}_2 = r_2e_1 + \tilde{r}x_1 - x_1e_3 - e_2 + ue_1, \\ \dot{e}_3 = x_1e_2 - b_2e_3 - \tilde{b}z_1. \end{cases} \quad (3)$$

Then, the three-dimensional Lyapunov function is defined as  $V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \tilde{\sigma}^2 + \tilde{r}^2 + \tilde{b}^2)$ , which is finite and larger than zero. The time rate of change of  $V$  along trajectories is given by  $\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \tilde{\sigma}\dot{\tilde{\sigma}} + \tilde{r}\dot{\tilde{r}} + \tilde{b}\dot{\tilde{b}}$ . Adopting the ideas from the parameters identification technique, we choose the adaptive formula of parameters as

$$\begin{cases} \dot{\tilde{\sigma}} = -(y_1 - x_1)e_1, \\ \dot{\tilde{r}} = -x_1e_2, \\ \dot{\tilde{b}} = z_1e_3, \end{cases} \quad (4)$$

which govern the evolution of estimation parameters. Simultaneously,  $\dot{V}$  reduces to be

$$\dot{V} = -\sigma_2e_1^2 + \sigma_2e_2e_1 + r_2e_1e_2 - e_2^2 + ue_1e_2 - b_2e_3^2. \quad (5)$$

With the simplest choice  $u = -(\sigma_2 + r_2)$ , the time rate of change of  $V$  turns out to be

$$\dot{V} = -\sigma_2e_1^2 - e_2^2 - b_2e_3^2. \quad (6)$$

In other words, the error dynamics are globally asymptotically stable at the origin when  $\sigma_2 > 0$  and  $b_2 > 0$ . Even though the estimated parameters would evolve with time passing, the three chaotic parameters are larger than zero in Lorenz system. Therefore, we can restrict the parameters to be positive during evolution to ensure  $\dot{V} < 0$ . Then the synchronization of uncertain chaotic systems is accomplished.

RK4 method is used to all of our simulations with time step being equal to 0.0001. We select the parameters of the master system as  $\sigma_1 = 10$ ,  $r_1 = 28$ ,  $b_1 = 2.67$  to ensure the chaotic behavior. The initial values are  $x_1(0) = 1$ ,  $y_1(0) = 0$ ,  $z_1(0) = 1$  and  $x_2(0) = 2$ ,  $y_2(0) = -1$ ,  $z_2(0) = -2$ . The parameters of response system start from  $\sigma_2(0) = 6$ ,  $r_2(0) = 12$  and  $b_2(0) = 5$ . The numerical results are illustrated in Figs. 1 and 2. Fig. 1 shows the errors between two chaotic systems. When the errors approach to zero, the synchronization of uncertain chaotic systems is realized. Fig. 2 shows the evolutions of  $\sigma_2(t)$ ,  $r_2(t)$ , and  $b_2(t)$ . Obviously, with time passing, the estimated parameters are able to approach the unknown parameters.

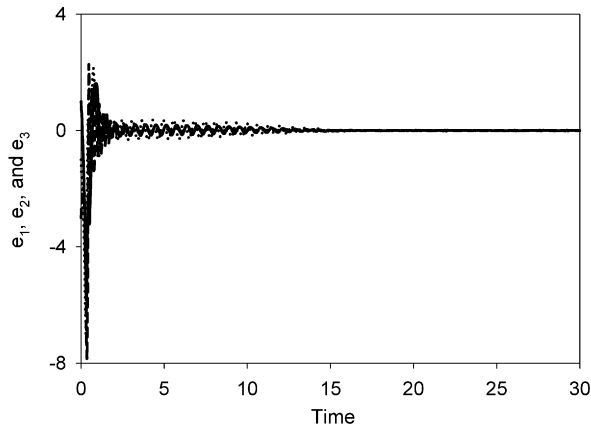


Fig. 1. The diagram presents the errors  $e_1(t)$ ,  $e_2(t)$  and  $e_3(t)$  between two coupled Lorenz systems.  $e_1(t) = x_2(t) - x_1(t)$  is labeled as a solid line,  $e_2(t) = y_2(t) - y_1(t)$  is labeled as a dotted line and  $e_3(t) = z_2(t) - z_1(t)$  is labeled as a dash line.

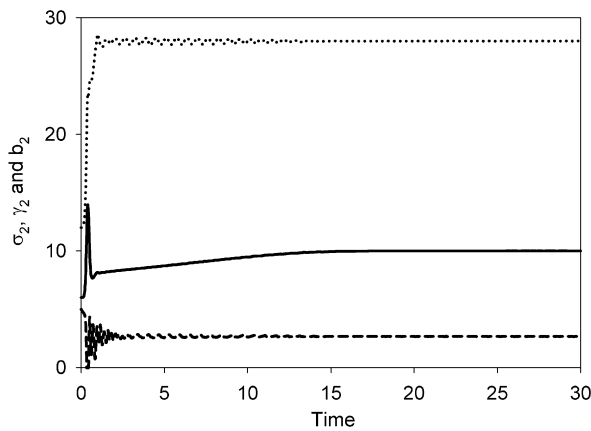


Fig. 2. The diagram presents the parameters  $\sigma_2(t)$ ,  $r_2(t)$  and  $b_2(t)$  of the slave Lorenz systems.  $\sigma_2(t)$  is labeled as a solid line,  $r_2(t)$  is labeled as a dotted line and  $b_2(t)$  is labeled as a dash line.

### 3. Chen system

In this section, we take Chen system into consideration. In 1999, Chen found another chaotic attractor, which is similar but topologically different from Lorenz system [26]. Due to its complex topological features, Chen system is relatively difficult to be controlled as compared to Lorenz system, especially its rapid change in  $z$ -direction [27]. The non-linear differential equations that describe the Chen system are

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz, \end{cases} \quad (7)$$

where  $a > 0$ ,  $b > 0$ ,  $c > 0$  and  $c < a < 2c$ .

Now, we consider the synchronization of uncertain Chen systems via unidirectional coupling.

The master system:

$$\begin{cases} \dot{x}_1 = a_1(y_1 - x_1), \\ \dot{y}_1 = (c_1 - a_1)x_1 - x_1z_1 + c_1y_1, \\ \dot{z}_1 = x_1y_1 - b_1z_1, \end{cases} \quad (8)$$

and  $a_1$ ,  $b_1$ , and  $c_1$  are fixed and unknown.

The slave system:

$$\begin{cases} \dot{x}_2 = a_2(y_2 - x_2), \\ \dot{y}_2 = (c_2 - a_2)x_2 - x_1z_2 + c_2y_2 + u(y_2 - y_1), \\ \dot{z}_2 = x_1y_2 - b_2z_2, \end{cases} \quad (9)$$

$a_2, b_2, c_2$  are the estimated parameters and  $u$  is the coupling variable. Similarly, we define  $\tilde{a} = a_2 - a_1, \tilde{b} = b_2 - b_1, \tilde{c} = c_2 - c_1, e_1 = x_2 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1$ , and the error dynamics can be decided as

$$\begin{cases} \dot{e}_1 = a_2(e_2 - e_1) + \tilde{a}(y_1 - x_1), \\ \dot{e}_2 = (c_2 - a_2)e_1 + (\tilde{c} - \tilde{a})x_1 - x_1e_3 + c_2e_2 + \tilde{c}y_1 + ue_2, \\ \dot{e}_3 = x_1e_2 - b_2e_3 - \tilde{b}z_1. \end{cases} \quad (10)$$

Let  $V$  be  $\frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2)$ , and the time rate of change of  $V$  becomes  $\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}}$ . Thus, the parameters adjustment equation can be chosen as

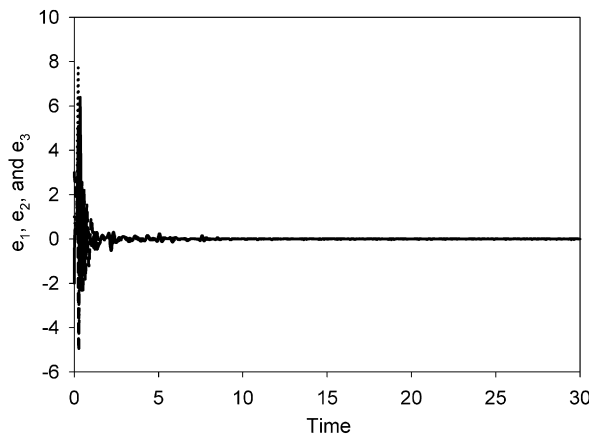


Fig. 3. The diagram presents the synchronization errors  $e_1(t), e_2(t)$  and  $e_3(t)$  of the unidirectional coupled Chen systems.  $e_1(t)$  is labeled as a solid line,  $e_2(t)$  is labeled as a dotted line and  $e_3(t)$  is labeled as a dash line.

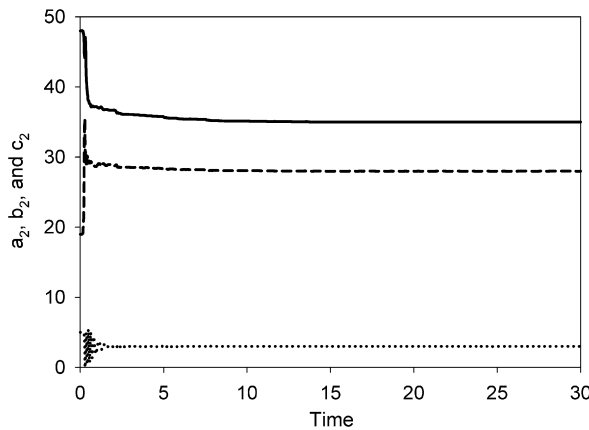


Fig. 4. The diagram presents the parameters  $a_2(t), b_2(t)$  and  $c_2(t)$  of the slave Lorenz systems.  $a_2(t)$  is labeled as a solid line,  $b_2(t)$  is labeled as a dotted line and  $c_2(t)$  is labeled as a dash line.

$$\begin{cases} \dot{a} = x_1 e_2 - (y_1 - x_1) e_1, \\ \dot{b} = z_1 e_3, \\ \dot{c} = -e_2(x_1 + y_1). \end{cases} \tag{11}$$

Then the time rate of change of  $V$  will become

$$\dot{V} = -a_2 e_1^2 + c_2 e_1 e_2 + (c_2 + u) e_2^2 - b_2 e_3^2. \tag{12}$$

If  $u = -\frac{c_2^2 + 4a_2 c_2}{4a_2} - k^2$  ( $k > 0$  and  $k \in R$ ), one has  $-a_2 e_1^2 + c_2 e_1 e_2 + (c_2 + u) e_2^2 < 0$  because the parameters of Chen system are positive. Therefore, when the errors between two systems are unequal to zero, the Lyapunov function is asymptotically stable at the origin. In other words, the synchronization can be achieved successfully.

The parameters of the master system are selected as the typical value,  $a_1 = 35$ ,  $b_1 = 3$  and  $c_1 = 28$ . The initial conditions are  $x_1(0) = 1$ ,  $y_1(0) = 0$ ,  $z_1(0) = -1$  and  $x_2(0) = -1$ ,  $y_2(0) = 1$ ,  $z_2(0) = 2$ . The parameters of the response system start from  $a_2(0) = 48$ ,  $b_2(0) = 5$ ,  $c_2 = 19$ , and constant  $k = 2$ . The results of synchronization and parameters identification are presented in Figs. 3 and 4, respectively.

#### 4. Effects of external noise

In real physical systems, it is impossible to neglect the effect of external noise. Because the chaotic system depends sensitively on a slight perturbation of signals, we have to ensure the synchronization and parameters identification are robust to the external noise. In this manuscript, we focus on the Lorenz case. Reconsidering the drive system

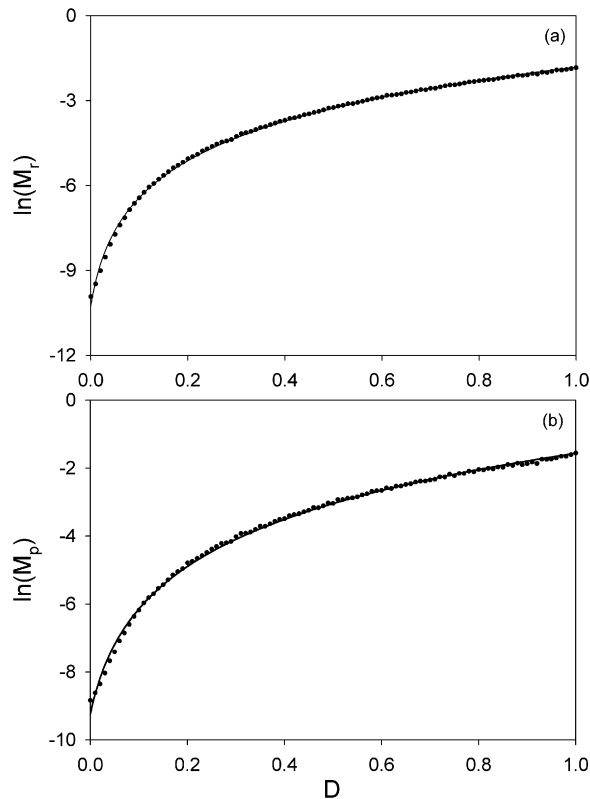


Fig. 5. (a) Illustrates the relationship between  $\ln(M_r)$  and noise intensity. (b) Illustrates the relationship between  $\ln(M_p)$  and noise intensity.

$$\begin{cases} \dot{x}_1 = \sigma_1(y_1 - x_1), \\ \dot{y}_1 = r_1x_1 - x_1z_1 - y_1 + \zeta(t), \\ \dot{z}_1 = x_1y_1 - b_1z_1, \end{cases} \quad (13)$$

where  $\zeta(t)$  is the Gaussian noise satisfying  $\langle \zeta(t) \rangle = 0$  and  $\langle \zeta(t)\zeta(t') \rangle = 2D\delta(t - t')$  in which  $D$  is the noise intensity. Here,  $\langle \dots \rangle$  denotes the time average. Naturally, we evaluate mean-square error between the response system  $(x_2, y_2, z_2)$  and the drive system  $(x_1, y_1, z_1)$  (denoted by  $M_r$ ) after controlling. For parameters identification technique, it is also important to calculate mean-square error between the parameters of two systems ( $M_p$ ). The definition of  $M$  is

$$M = \frac{1}{T} \int_0^T [S_1(t) - S_2(t)]^2 dt. \quad (14)$$

Fig. 5(a) and (b) shows the results of numerical simulation. We calculate mean-square errors under the condition used in Section 2. In order to diminish the effect of transient signal, we start to calculate the mean-square error after sufficient iteration times ( $2 \times 10^6$ ). The time step  $T$  is  $5 \times 10^5$ . We calculate the values over 100 times and average them to eliminate fluctuation. Fig. 5(a) illustrates the relationship between  $\ln(M_r)$  and noise intensity while (b) illustrates the relationship between  $\ln(M_p)$  and noise intensity. The relationship can be formularized as

$$\begin{cases} M_r = (D - r_1)^{r_2} \cdot e^{r_3}, \\ M_p = (D - p_1)^{p_2} \cdot e^{p_3}, \end{cases} \quad (15)$$

where  $r_1 = -0.02$ ,  $r_2 = 2.16$ ,  $r_3 = -1.86$  and  $p_1 = -0.03$ ,  $p_2 = -2.24$ ,  $p_3 = -1.64$ .

In Eq. (15),  $M_r$  and  $M_p$  become larger with the increase of noise density and follow two different power laws when  $D \leq 1.0$ . However, the results are acceptable in real applications. In other words, the techniques are robust to external noise.

## 5. Conclusion

Based upon the simple control method and parameters identification technique, we synchronize the response system to be the drive system. By using the Lorenz and the Chen systems, the chaotic synchronization is achieved exactly. In addition, the noise analysis is under our discussion to ensure the validity in real physical systems. In fact, the electric Lorenz circuit is useful in the field of secure communication, and the techniques may make it more applicable in such field.

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