Distinguishing a driver from a response system

Shou L. Bu,[1,2,a)] I-Min Jiang,[1] and Ming C. Ho[3]
[1]*Department of Physics, National Sun Yat-sen University, 804 Kaohsiung, People's Republic of China*
[2]*School of Physics and Optoelectronic Technology, Dalian University of Technology, 116024 Dalian, People's Republic of China*
[3]*Department of Physics, National Kaohsiung Normal University, 804 Kaohsiung, People's Republic of China*

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This study presents an approach for distinguishing a driver from a response system. The proposed method can be applied to both unidirectional or bidirectional interactions, and to identical or structurally different systems. Compared with most previously proposed schemes, the present method is so "simple" that the driver-response relationships can generally be detected using a direct graphic way. On the other hand, quantitative estimation is also developed using the idea of the correlation dimension. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2896093]

A major issue in nonlinear dynamics and nonlinear time series analysis is the interplay among complex dynamic systems. Of particular interest is how to assess the interaction between two systems via analyzing the interrelation between their output signals. Up to now, many previously proposed schemes have either only been applicable to the generalized synchronization regime, or have had difficulty assessing the coupling direction in structurally different systems. In particular, all suggested schemes are so "complicated" that even for the simplest case, i.e., the case involving unidirectionally coupled identical systems, large-scale computations are required to distinguish driver-response relationships. This study suggests a method of addressing this problem. The proposed method can be applied to both weak and strong interactions, and to both identical and structurally different systems. Furthermore, the proposed method is sufficiently "simple" that, in most cases, it is even possible to directly estimate the driver-response relationships graphically. Additionally, a systematic technique based on the correlation dimension is devised to quantify the coupling asymmetry.

I. INTRODUCTION

Recently, considerable attention has been devoted to detecting driver-response relationships between two interacting systems, and various schemes have been designed to address the problem.[1–4] One of these schemes exploits the idea of generalized synchronization of unidirectionally coupled systems, and detects the direction of coupling by examining the correspondence between neighbors in the embedding spaces of the driver and response.[1,2,5] Another of these schemes, based on the notion of phase synchronization[6] of coupled systems, uses the familiar result that weak interaction usually first alters the phases of the state variables and then their amplitudes, and thus proposes analyzing the relationship between the phases of the oscillators to estimate the direction and strength of coupling.[3] Additionally, for strongly coupled systems, schemes based on information theory are also developed to achieve such goal.[4,7] More recently, Romano *et al.* introduced a method for detecting and quantifying the asymmetry of the coupling between interacting systems based on their recurrence properties.[8] However, most schemes are either only applicable in the generalized synchronization regime, or have difficulty assessing the coupling direction in structurally different systems. Moreover, all schemes proposed to date are so "complicated" that even for the simplest case, i.e., unidirectionally coupled identical systems, there is no simple way to distinguish driver from response relationships directly.

This work suggests an approach to reveal driver-response relationships between interacting systems. The proposed method can be applied to both weak and strong interactions, and to identical or structurally different systems. Furthermore, the proposed method is sufficiently "simple" that, as will be demonstrated below, in most cases it is possible to directly distinguish the coupling direction graphically. On the other hand, a systematic technique is also developed to quantify the degree of asymmetry of the interaction.

The outline of this paper is as follows. Section II first introduces the basic idea of the proposed method, and then a graphic way is presented in Sec. II A. Furthermore, in Sec. II B, a quantitative description is developed by exploiting the fractal correlation dimension. Finally, conclusions are given in Sec. III.

II. THE APPROACH

The proposed approach is based on the observation that the topological properties of the attractor set derived from a so-called return map[9] systematically change when coupling is imposed on the underlying system. As demonstrated by Lorenz,[9] by following one of the variables (or their linear or nonlinear combinations, i.e., a time series generated from the state variables, as this study does) of an oscillator, it is pos-

[a)]Author to whom correspondence should be addressed. Electronic mail: busl1972@alumni.nsysu.edu.tw.

sible to produce a return map where the dynamics are attracted to an almost one-dimensional set. Following this lead, we can construct the return map from the time series, e.g., $x_i(t)$, as follows. Starting from some arbitrary point in time, define t_n as the time when $x_i(t)$ reaches its nth (local) maximum, and X_n as the value of x at that moment. The return map X_{n+1} *vs* X_n can thus be constructed. For an isolated oscillator, the map has an attractor that looks almost one-dimensional. However, it is observed that, for a response system, the fractal dimension is clearly biased away from being one-dimensional except in situations of complete synchronization. Indeed, the attractor exceeds one dimension even in the case of very weak interactions. This phenomenon occurs because the driven term can generally be seen as stochastic disturbing with respect to the response system and thus causes dispersion of the considered attractor set. In this work, this study exploits the concept of correlation dimension[10] to quantify the observed fact. As compared with most previous schemes, the driver-response relationships can be directly uncovered using a graphic method in most cases. On the other hand, in situations involving very weak interaction or nearly symmetric coupling in bidirectional interaction, to assess the interaction between the two systems, it is necessary to calculate and compare the correlation dimen-

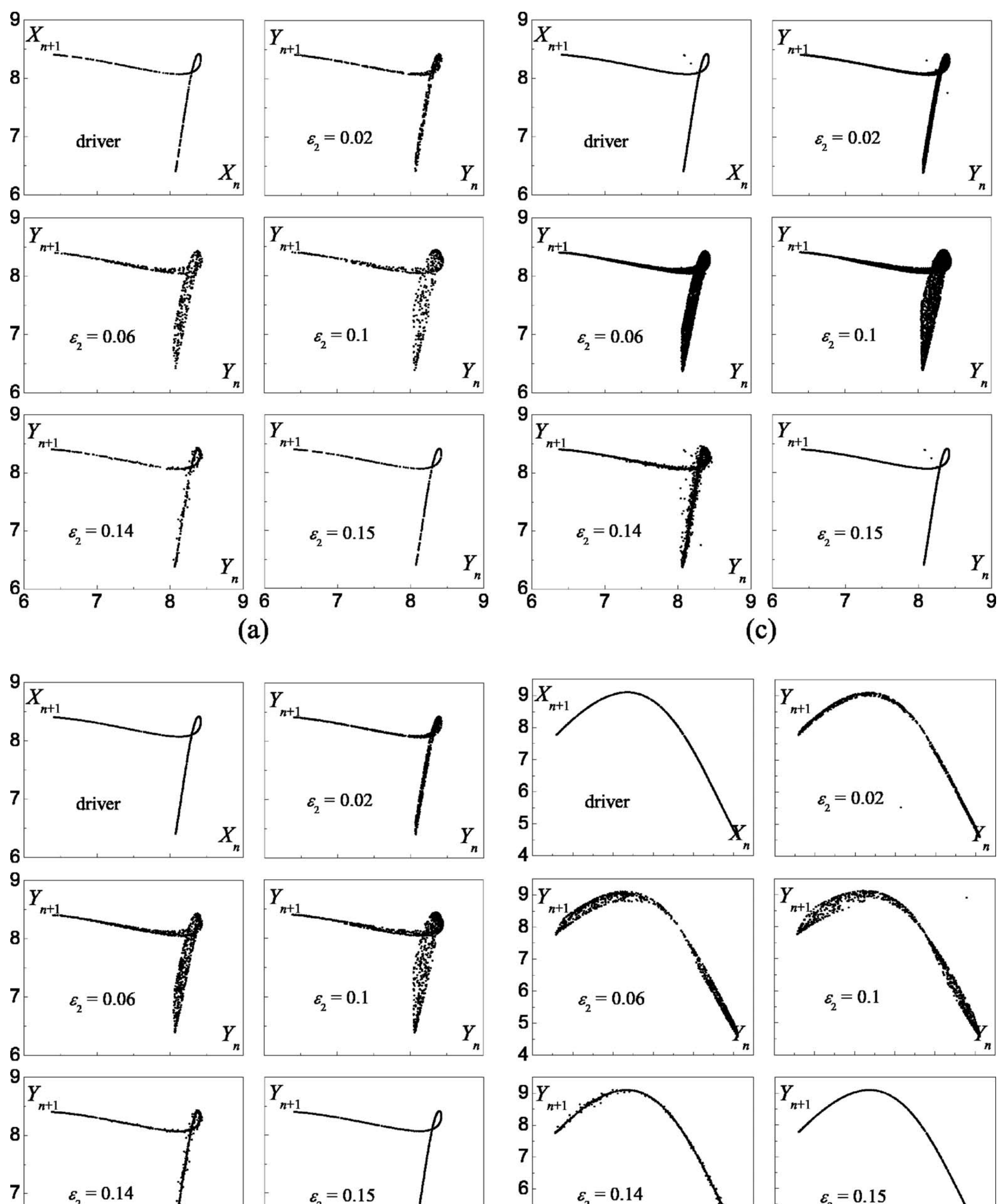

FIG. 1. Illustrations of attractors $X_{n+1} \sim X_n$ and $Y_{n+1} \sim Y_n$ corresponding to the driver and response system for unidirectional coupling with $\varepsilon_2=0.02$, 0.06, 0.1, 0.14, and 0.15. In (a)–(c), the time series $s_x(t)=x_1(t)+x_2(t)$ and $s_y(t)=y_1(t)+y_2(t)$, while in (d) $s_x(t)=x_1(t)$ and $s_y(t)=y_1(t)$. Additionally, the number of data points is (a) $N=500$, (b) $N=1000$, (c) $N=5000$, and (d) $N=1000$, respectively.

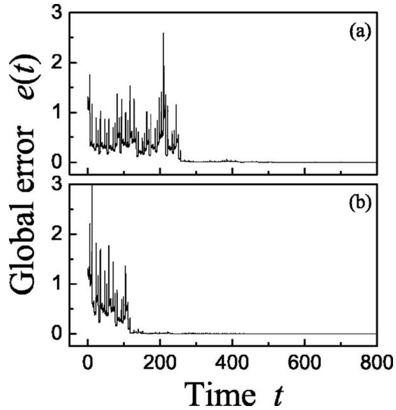

FIG. 2. The global error $e(t)$ (as defined in the text) vs time t for (a) $\varepsilon_1=0$, $\varepsilon_2=0.155$ and (b) $\varepsilon_1=0.03$, $\varepsilon_2=0.13$ is presented here.

sions of the attractors with respect to the interacting systems. The following first presents a graphic way, and then introduces the concept of the correlation dimension and applies it to the problem considered here.

A. Graphic way

Specifically, first consider two coupled chaotic Rössler oscillators as follows:

$$\dot{x}_1=-x_2-x_3+\varepsilon_1(y_1-x_1),\quad \dot{x}_2=x_1+0.2x_2,$$
$$\dot{x}_3=0.2+x_3(x_1-4.5), \tag{1}$$

$$\dot{y}_1=-y_2-y_3+\varepsilon_2(x_1-y_1),\quad \dot{y}_2=y_1+0.2y_2,$$
$$\dot{y}_3=0.2+y_3(y_1-4.5), \tag{2}$$

where ε_1 and ε_2 denote the coupling strengths. Now, two interaction types are studied.

(i) Unidirectional interaction, without loss of generality, letting $\varepsilon_1=0$ and $\varepsilon_2>0$. Figures 1(a)–1(c) illustrate the attractors $X_{n+1}\sim X_n$ and $Y_{n+1}\sim Y_n$, where X_n and Y_n give the nth local maxima of time series $s_x(t)=x_1(t)+x_2(t)$ and $s_y(t)=y_1(t)+y_2(t)$. The coupling strengths $\varepsilon_2=0.02$, 0.06, 0.1, 0.14, and 0.15 are explicitly labeled in the figure, and the numbers of data points used are (a) $N=500$, (b) $N=1\times10^3$, and (c) $N=5\times10^3$, respectively. The figure reveals that, as the coupling strength $\varepsilon_2$increases, the attractors for the response system expand and are biased away from one dimension. On the other hand, as ε_2 increases further, the attractors of the response system approach the attractor of the driver system again, as illustrated in Figs. 1(a)–1(c) for $\varepsilon_2=0.14$ and 0.15. The reason is that, if ε_2 exceeds some critical value ε_c, i.e., $\varepsilon_2\geq\varepsilon_c$, complete synchronization is achieved between the considered systems. To illustrate, the global error $e(t)=\sqrt{\sum_{i=1}^{n}(y_i(t)-x_i(t))^2}$ versus time t for $\varepsilon_1=0$ and $\varepsilon_2=0.155$ is plotted in Fig. 2(a). Notably, the method presented in this study does not depend on the specific observation selected. For illustration, in Fig. 1(d), the corresponding attractors generated from time series $x_1(t)$ and $y_1(t)$ are presented, where $N=1\times10^3$ data points are plotted and the coupling strengths are $\varepsilon_2=0.02$, 0.06, 0.1, 0.14, and 0.15, respectively. The driver-response relationships between X and Y systems are found to be equally clearly distinguished.

(ii) Bidirectional interaction, namely $\varepsilon_1>0$ and $\varepsilon_2>0$. In this situation, the first coupling coefficient is held to be $\varepsilon_1=0.03$, and the other one is ε_2, and is adjusted from 0.01 to 0.13. In Fig. 3, the attractors $X_{n+1}\sim X_n$ and $Y_{n+1}\sim Y_n$ corresponding to system (1) and (2) are shown for $\varepsilon_1=0.03$ and various values of ε_2, where $N=1\times10^3$ data points are used and the first and third column list $X_{n+1}\sim X_n$, and the second and fourth column list $Y_{n+1}\sim Y_n$, respectively. Also, X_n and Y_n denote the nth local maxima for time series $s_x(t)=x_1(t)+x_2(t)$ and $s_y(t)=y_1(t)+y_2(t)$. First, for $\varepsilon_1=0.03$ and $\varepsilon_2=0.01$, it can be seen that the attractor for $X_{n+1}\sim X_n$ expands wider than that for $Y_{n+1}\sim Y_n$. Subsequently, for $\varepsilon_1=\varepsilon_2=0.03$, the two attractors all become sparsely contracted sets owing to couplings. Given $\varepsilon_2=0.05$, 0.07, and 0.09, it can be seen that the region of attractors for $Y_{n+1}\sim Y_n$ exceeds

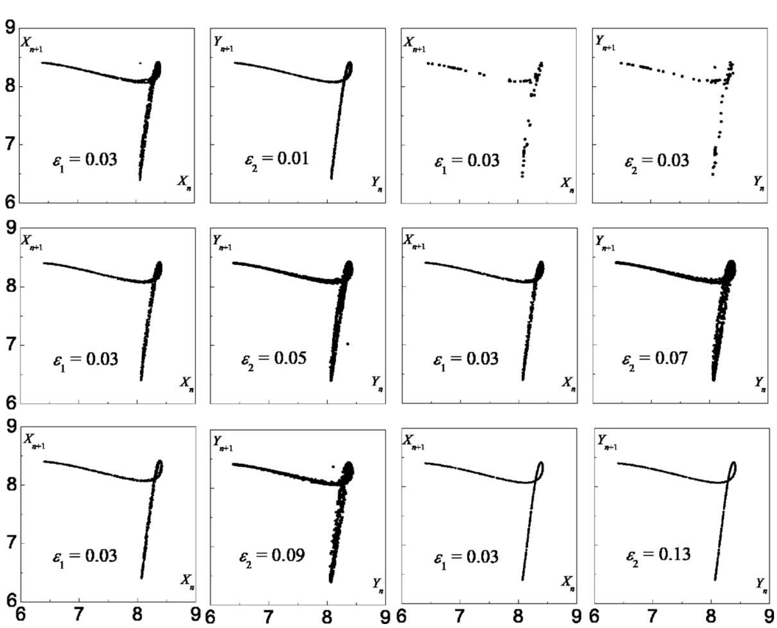

FIG. 3. Illustrations of the attractors $X_{n+1}\sim X_n$ and $Y_{n+1}\sim Y_n$ corresponding to two bidirectionally coupled systems governed by Eqs. (1) and (2) for $\varepsilon_1=0.03$ and various values of ε_2, where the first and third columns list $X_{n+1}\sim X_n$, and the second and fourth columns list $Y_{n+1}\sim Y_n$, respectively. Here, $N=1000$, and $s_x(t)=x_1(t)+x_2(t)$ and $s_y(t)=y_1(t)+y_2(t)$ are taken.

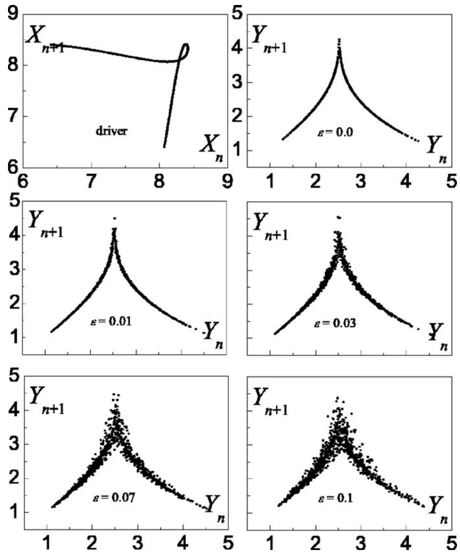

FIG. 4. Illustrations of the attractors $X_{n+1} \sim X_n$ and $Y_{n+1} \sim Y_n$ corresponding to two structurally different systems governed by Eqs. (3) and (4). Here, $N=1000$, $s_x(t)=x_1(t)+x_2(t)$, $s_y(t)=0.01y_1(t)y_2(t)$, and the coupling strengths $\varepsilon=0.0$, 0.01, 0.03, 0.07, and 0.1 are taken.

that for $X_{n+1} \sim X_n$ as illustrated in the figure. For larger values of ε_2, complete synchronization between the two coupled systems is achieved, and the attractors thus approach each other as shown in the figure for $\varepsilon_1=0.03$ and $\varepsilon_2=0.13$. Figure 2(b) presents the global error $e(t)$ versus time t for $\varepsilon_1=0.03$ and $\varepsilon_2=0.13$.

Furthermore, we consider the more challenging case involving two coupled structurally different system,[1] in which the driver is an autonomous Rössler system with

$$\dot{x}_1 = -x_2 - x_3, \quad \dot{x}_2 = x_1 + 0.2x_2, \quad \dot{x}_3 = 0.2 + x_3(x_1 - 4.5), \tag{3}$$

which drives a Lorenz system in which the equation for $\dot{y}_2$ is augmented via a driving term involving x_2,

$$\begin{aligned} &\dot{y}_1 = 10(-y_1 + y_2), \quad \dot{y}_2 = 28y_1 - y_2 - y_1y_3 + \varepsilon x_2^2, \\ &\dot{y}_3 = y_1y_2 - \tfrac{8}{3}y_3. \end{aligned} \tag{4}$$

The unidirectional coupling is introduced in the final term of the second equation, the constant ε being the coupling strength.

Figure 4 shows the attractors corresponding to driver Rössler and response Lorenz systems for time series $s_x(t) = x_1(t)+x_2(t)$ for $s_y(t)=0.01y_1(t)y_2(t)$ with coupling strengths $\varepsilon=0.0$, 0.01, 0.03, 0.07, and 0.1, respectively, where $N=1 \times 10^3$ data points are employed. It is found that as ε increases, the attractor corresponding to the response system clearly becomes biased away from one dimension.

In short, to detect the direction (or asymmetry) of the coupling between two systems X and Y, it is possible to plot and compare the corresponding attractors $X_{n+1} \sim X_n$ and $Y_{n+1} \sim Y_n$, and identify the response system with the expanding one. On the other hand, it is possible to quantitatively describe the observed phenomena by calculating and comparing the correlation dimensions of the attractors. Specifically, the response system can be identified with the system with a larger correlation dimension as discussed below.

B. Quantitative description: Correlation dimension

To accurately describe the observed phenomena, we now calculate the correlation dimensions of the attractor sets as illustrated in Sec. II A for interacting systems. The concept of the correlation dimension was introduced by Grassberger and Procaccia,[10] and has been widely used to characterize chaotic attractors. To define the correlation dimension for an attractor including N points, $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, it is necessary to ask for the number of points lying within distance ρ of each point i, excluding point i itself. Next, the relative number is formally written as

$$n_i(\rho) = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \Theta(\rho - |\mathbf{x}_i - \mathbf{x}_j|), \tag{5}$$

where $\Theta(x)$ denotes the Heaviside step function satisfying $\Theta(x)=0$ if $x<0$, and $\Theta(x)=1$ if $x \geq 0$. Then, we compute the so-called correlation sum $C(\rho)$,

$$C(\rho) = \frac{1}{N}\sum_{i=1}^{N} n_i(\rho) = \frac{1}{N(N-1)} \sum_{j=1, j\neq i}^{N} \Theta(\rho - |\mathbf{x}_i - \mathbf{x}_j|). \tag{6}$$

Often the limit $N \to \propto$ is also added to assume that we characterize the entire attractor.[11] Finally, the correlation dimension D is defined as the number

$$D = \lim_{\rho\to 0} \frac{\log C(\rho)}{\log \rho}. \tag{7}$$

Since any real data set consists of a finite number of points, there is some minimum distance between points. When ρ is less than that minimum distance, the correlation sum $C(\rho)$ equals zero and so too does the correlation dimension. Therefore, what is done in practice is to compute $C(\rho)$ for some range of ρ values and then plot $\log C(\rho)$ as a function of $\log \rho$ as illustrated in Fig. 5(a). This study finds that, to obtain a reliable result for the correlation dimension, the number of data points should be no less than $N=1 \times 10^3$, and $N=3 \times 10^3$ is used in this study. Based on the observed phenomena described above, it is reasonable to consider that a higher correlation dimension with respect to the attractor $X_{n+1} \sim X_n$ or $Y_{n+1} \sim Y_n$ corresponds to the response system (or that driven by stronger coupling strength in bidirectional interaction cases). Consequently, the following computes and compares the correlation dimensions of attractors $X_{n+1} \sim X_n$ and $Y_{n+1} \sim Y_n$ to distinguish a driver from a response system.

First, reconsider two interacting Rössler oscillators governed by Eqs. (1) and (2). To calculate the correlation dimension D, $N=3 \times 10^3$ data points $\mathbf{x}_i = (X_{i+1}, X_i)$ [or (Y_{i+1}, Y_i)] for $i=1,2,\ldots,N-1$ and $\rho \in [10^{-3.5}, 10^{-1.75}]$ are taken, respectively, in the following simulations: X_i (or Y_i) is the ith maximum of the observed time series. Additionally, all the results given below are obtained by averaging over 100 different trajectories. For example, $\log C(\rho)$ versus $\log \rho$ for attractor $Y_{n+1} \sim Y_n$ with $\varepsilon_1=0.03$ and $\varepsilon_2=0.08$ is plotted in Fig. 5(a). The slope of the fitting line shown here, i.e.,D_y, is 1.794, and has standard deviation 0.008. As mentioned previously, this study holds the first coupling coefficient to be $\varepsilon_1=0.03$, and adjusts another coefficient ε_2 from 0.01 to 0.12. Figure 5(b)

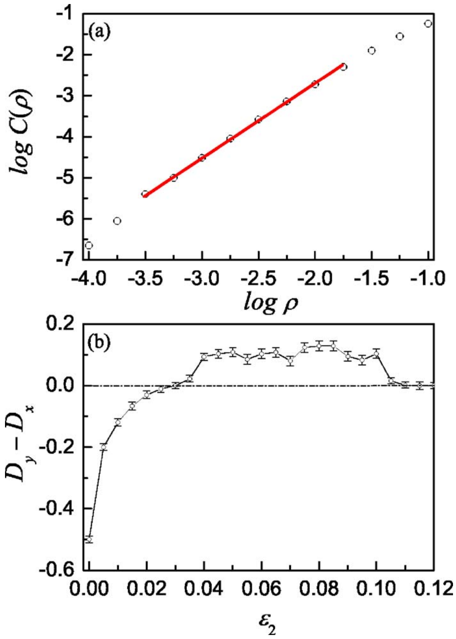

FIG. 5. (Color online) (a) $\log C(\rho) \sim \log \rho$ for attractor $Y_{n+1} \sim Y_n$ with $\varepsilon_1 = 0.03$ and $\varepsilon_2 = 0.08$. The slope of the fitting line shown here, i.e., D_y, is 1.794, and has standard deviation 0.008. Here, $N=3000$, and $\rho \in [10^{-3.5}, 10^{-1.75}]$ are taken. (b) Plots of $(D_y - D_x) \sim \varepsilon_2$ with standard deviations, where $\varepsilon_1 = 0.03$ is fixed in the simulations. Furthermore, $\Delta D = D_y - D_x$ quantify the asymmetry in coupling of two bidirectionally coupled systems. The zero line is also plotted to facilitate orientation (dot-dashed line).

plots $(D_y - D_x) \sim \varepsilon_2$ with standard deviations. It can be seen that $\Delta D(\varepsilon_2) = D_y(\varepsilon_2) - D_x(\varepsilon_2)$ can accurately quantify the coupling asymmetry.

Next, similar results can be obtained for structurally different systems. For example, consider once again two coupled systems governed by Eqs. (3) and (4), where the Lorenz system is driven by the Rössler oscillator. This study plots $\Delta D = (D_L - D_R) \sim \varepsilon$ with coupling strength ε ranging from 0 to 0.095 in Fig. 6, where D_L and D_R are the correlation dimensions[12] of the attractors $X_{n+1} \sim X_n$ and $Y_{n+1} \sim Y_n$ corresponding to the Lorenz and Rössler systems, respectively. $\Delta D(\varepsilon)$ is found to increase with ε indicating the causal relationship between the two interacting systems very well.

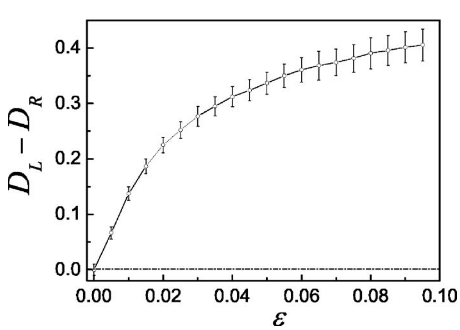

FIG. 6. Plots of $(D_L - D_R) \sim \varepsilon$ with standard deviations and the coupling strength ε range from 0 to 0.095, where D_L and D_R are correlation dimensions of the attractors $X_{n+1} \sim X_n$ and $Y_{n+1} \sim Y_n$ corresponding to the Lorenz and Rössler systems, respectively. The zero line is also plotted for orientation (dot-dashed line).

A related issue concerns the so-called passive experiments[8,13] in which the coupling strength between two systems cannot be varied systematically. As noted by Romano *et al.*,[8] this is the case in numerous situations. Thiel *et al.*[13] proposed a *twin surrogates* test to resolve this problem. Regarding the method developed in this study, it is found that, when $|\varepsilon_2 - \varepsilon_1|$ is very small (including the unidirectional and bidirectional cases discussed above), $|\Delta D|$ is nearly zero. Therefore, to assess the statistical significance of the results, it is necessary to generate surrogates.

III. CONCLUSIONS

In sum, this study proposes an approach for identifying the direction of coupling between two interacting systems. The proposed approach can be applied to both unidirectional coupling and bidirectional coupling, and for weak and strong couplings. Furthermore, the proposed approach can detect not only driver-response relationships between identical systems, but also structurally different systems. Particularly, according to the proposed scheme, in most cases it is even possible to distinguish a driver from a response system graphically.

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