

Chaotic behavior in the disorder cellular automata

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Abstract

Disordered cellular automata (DCA) represent an intermediate class between elementary cellular automata and the Kauffman network. Recently, Rule 126 of DCA has been explicated: the system can be accurately described by a discrete probability function. However, a means of extending to other rules has not been developed. In this investigation, a density map of the dynamical behavior of DCA is formulated based on Rule 22 and other totalistic rules. The numerical results reveal excellent agreement between the model and original automata. Furthermore, the inhomogeneous situation is also discussed.

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1. Introduction

In recent decades, cellular automata (CA) have attracted considerable interest [1–4]. Besides the ability to exhibit very complex behavior based on relatively simple rules, CA are good models of a wide variety of physical systems [5,6], including magnetization in solids [7], reaction–diffusion processes [8,9], fluid dynamics for complex situations [10], growth phenomena [11,12], traffic flow models [13,14] and others. Hence, the dynamics of CA must be investigated to realize and predict the behavior in real systems.

CA are generally described in terms of two concepts – configuration and evolution rule. Different rules capture various evolution patterns. Wolfram grouped all the elementary cellular automata (ECA) into four classes: homogeneous state (class I), separated simple stable or periodic structures (class II), chaotic pattern (class III) and complex localized structures (class IV). Most ECA can be used to simulate one-dimensional growth and flow. However, when spatial location becomes less meaningful in the system, as in spin models with a disordered long-range interaction and the problem of cell differentiation [15,16], Kauffman networks (KNs) are more effective than ECA at modeling the systems. In fact, ECA can be considered to be a KN with $k = 3$ connections, but with an exclusive evolution rule and deterministic connection. The dynamics of KNs have been extensively studied, and the details referred to Refs. [17–19,25].

This study addresses the intermediate class between ECA and KNs. Such intermediate automata are called disordered cellular automata (DCA) or random Boolean networks, whose connections are still chosen randomly but whose

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evolution rules are identical for all cells (nodes) [20]. The dynamics of DCA are still far less understood than those of ECA and KNs. The generalized Rule 126 applied in DCA can be accurately described by a density map, which has recently been explicated [21]. Matache and Heidel lately provided a generalized formulation to model the inhomogeneous DCA based on Rule 126 [22]. Like Rule 126, Rule 22 is another legal, totalistic and complex rule among the 256 rules. The evolution rule is expressed as the Boolean function, $c_n(t + 1) = f_n[c_{n-1}(t), c_n(t), c_{n+1}(t)]$, where n is the site index of each cell. Rule 22 then becomes $f[0, 0, 1] = f[0, 1, 0] = f[1, 0, 0] = 1$, and $f = 0$ for the remaining five possible situations. It naturally extends what has been done based on Rule 22. However, the method of so doing has not yet been determined [22]. Accordingly, the aim of this report is to formulate the density maps to describe the dynamical behavior of DCA based on Rule 22. Furthermore, the generalized situation is considered and the idea extended to other totalistic rules. The inhomogeneous situation (in which the numbers of connections of cells differ) is also discussed.

2. Density map of DCA

Consider a disorder cellular automaton with N cells ($N \rightarrow \infty$). Each cell c_n , where $n = 1, 2, \dots, N$, is described by 1 or 0. The connections of a cell c_n are assigned randomly from other ones in the identical network, and the number of connections is denoted by k , which is fixed for all cells during evolution. Suppose the states of cells are updated simultaneously, and all are governed by Rule 22 from discrete time t to $t + 1$. The evolution rule is as follows:

$$c_n(t + 1) = \begin{cases} 1 & c_n(t) + S_{nk}(t) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $S_{nk} = \sum_{j=1}^k c_{nj}(t)$ is the sum of the k random connections of cell c_n . In other words, only a single cell is in state 1, and the others are in state 0 among c_n and its connections. Then, c_n will be 1 next iteration. Otherwise, c_n becomes 0.

The density (the probability that a cell is in state 1) for the DCA at time t is $p(t) = N^{-1} \sum_{n=1}^N c_n(t)$. Of course, the density also satisfies $p(t) = N_1 / (N_0 + N_1)$, where N_1 is the number of cells in state 1 and N_0 is the number of cells in state 0 at time t . Now, we start with the derivation of $N_{1 \rightarrow 1}(t)$ which denote that the number of cells are 1 at time t and remain 1 at time $t + 1$. From Eq. (1), when $c_n(t) = 1$, $c_n(t + 1) = 1$ if k connections are all in state 0. The formulation is,

$$N_{1 \rightarrow 1}(t) = N_1 [1 - p(t)]^k. \tag{2}$$

Similarly, the number of cells whose states are changed from 1 to 0 is

$$N_{1 \rightarrow 0}(t) = N_1 \{1 - [1 - p(t)]^k\}. \tag{3}$$

Under the condition $c_n(t) = 0$, one (and only one) of the connections must be 1 while the others are 0; then $c_n(t + 1)$ becomes 1. Therefore,

$$N_{0 \rightarrow 1}(t) = N_0 \binom{k}{1} p(t) [1 - p(t)]^{k-1} = N_0 k p(t) [1 - p(t)]^{k-1} \tag{4}$$

and

$$N_{0 \rightarrow 0}(t) = N_0 \{1 - k p(t) [1 - p(t)]^{k-1}\}. \tag{5}$$

The above equations must fulfill the normalization condition:

$$N_{1 \rightarrow 1}(t) + N_{1 \rightarrow 0}(t) + N_{0 \rightarrow 1}(t) + N_{0 \rightarrow 0}(t) = N.$$

The quality $p(t + 1) = N^{-1} [N_{1 \rightarrow 1} + N_{0 \rightarrow 1}]$ which represents the probability of finding out a cell in state 1 at time $t + 1$ can be constructed. Inserting the results from Eqs. (2) and (4) into $p(t + 1)$ yields:

$$P(t + 1) = f(p(t)) = (1 + k)p(t)[1 - p(t)]^k. \tag{6}$$

The formulation is the density map of DCA based on Rule 22.

The map must be verified to be consistent with the original system. With $N = 10^3$ and $k = 12$, the first two iterations of the model and the original network are presented. Fig. 1a and b plot $p(t + 1)$ and $p(t + 2)$ versus $p(t)$ individually. The numerical results of the map are represented by the solid line and the results of the original automata are presented as points. They exhibit excellent agreement. Of course, as N ($N \rightarrow \infty$) increases, the agreement will be approved further.

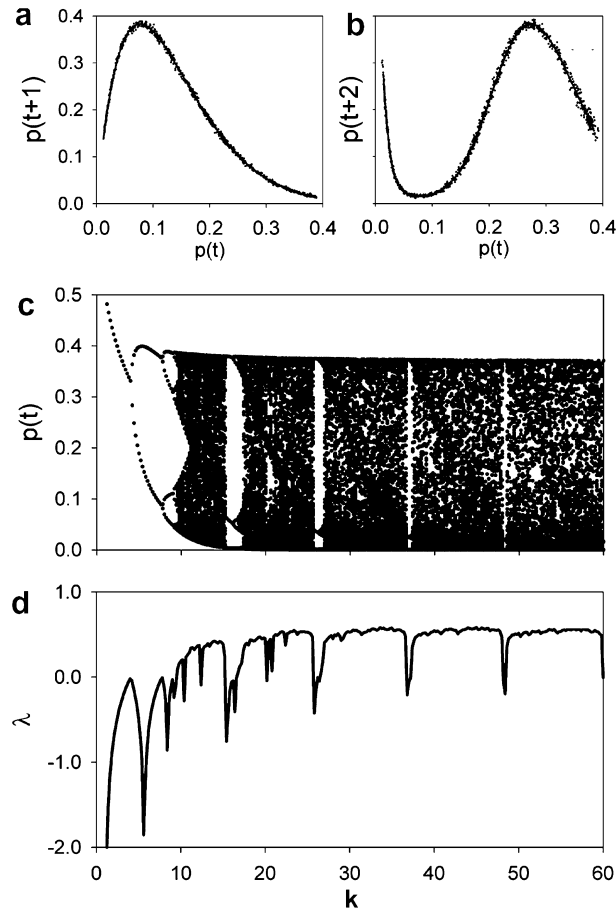


Fig. 1. Simulation results of DCA based on Rule 22: (a,b) first two iterations $P(t + 1)$ and $P(t + 2)$ versus $P(t)$ with $k = 12$; numerical results of the map are represented by the solid line and the results of the original automata are represented by points, (c) the bifurcation diagram when $k \in R$, and (d) the dependence of the maximal Lyapunov exponent λ on the parameter k .

With $k = 2$, the evolution rule returns to the original definition in ECA, and the map has a stable fixed point $p^* = 1 - \sqrt{3}/3$, which is consistent with the results reported elsewhere [20]. In fact, period-1 is not the only solution to the map. Fig. 1c displays the bifurcation diagram when $k \in R$. As k increases, the density map processes the route to chaos via period-doubling bifurcations. The maximal Lyapunov exponent λ represents the exponential rate at which an arbitrarily small displacement is amplified, and $\lambda_{max} > 0$ suffices to ensure that the maps reveal chaos. Fig. 1d plots the dependence of the maximal Lyapunov exponent λ on the chaotic parameter k . A comparison with the bifurcation diagram clearly demonstrates the excellent connections.

Now, consider the generalized condition. Rewrite the evolution rule in Eq. (1) as

$$c_n(t + 1) = \begin{cases} 1 & c_n(t) + S_{nk}(t) = m, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where m is the code number of the rule and $m \leq k$. Similarly, the number of cells that remain unchanged in state 1 is

$$N_{1 \rightarrow 1}(t) = N_1 \binom{k}{m-1} p^{m-1}(t) [1 - p(t)]^{k-(m-1)}, \tag{8}$$

and the number of cells whose state changes from 0 to 1 is

$$N_{0 \rightarrow 1}(t) = N_0 \binom{k}{m} p^m(t) [1 - p(t)]^{k-m}. \tag{9}$$

Finally, the density map is

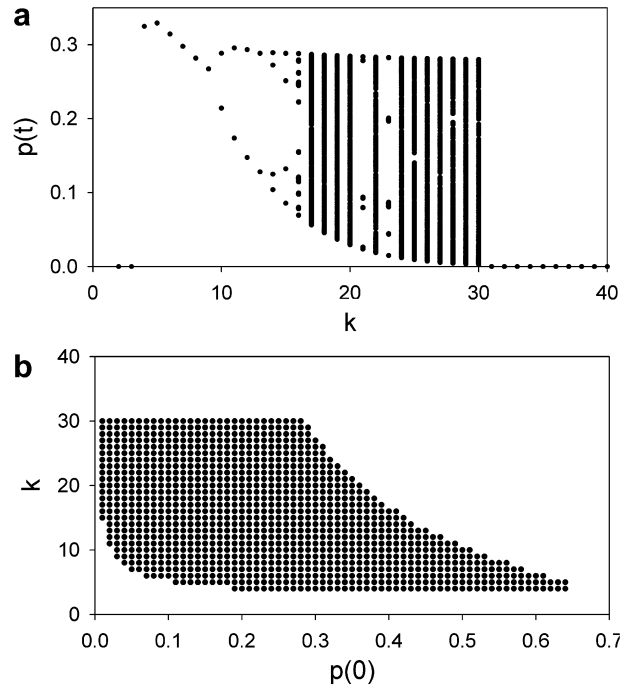


Fig. 2. (a) Bifurcation diagram with $p(0)=0.20$ and $m=2$ based on the density map in Eq. (10). In the bifurcating region $k \in [k_{\min}, k_{\max}] = [4, 30]$, the dynamics follow the route to chaos via period-doubling. (b) Effect of initial conditions $p(0)$. The black points indicate the area in which the system undergoes the bifurcation, and outside $p(t)$ converges to zero as time passes.

$$P(t + 1) = f(p(t)) = \left[\binom{k}{m-1} + \binom{k}{m} \right] p^m(t) [1 - p(t)]^{(k-m+1)}, \tag{10}$$

with parameters $k, m \in N$. For $m = 1$, the result is exactly the formulation obtained in Eq. (6). For $m = 2$, the map corresponds to the DCA based on Rule 104, which is another legal, totalistic evolution function [1]. Fig. 2a presents the bifurcation diagram with $p(0) = 0.20$ and $m = 2$. In the bifurcating region $k \in [k_{\min}, k_{\max}] = [4, 30]$, the dynamical behaviors undergo the periodic doubling route and become chaos with larger k . However, outside the region (including when $k = 2$ in which Eq. (7) is analogous to the original definition in ECA), the density $p(t)$ decreases to zero as time passes. The result verifies that Rule 104 is categorized as class II by Wolfram, although it also exhibits chaotic behaviors with suitably selected connection parameters as in Fig. 2a. Notably, the dynamics depends on the initial condition $p(0)$. Fig. 2b shows the numerical results. The black points display the area in which the system exhibits nonzero solutions; outside the area $p(t)$ converges to zero for all values of k .

3. Inhomogeneous situation and other totalistic rules

The number of connections of each cell is known not be identical in real systems. Therefore, the inhomogeneous situation must be considered [22]. Suppose G_j is the group of all cells that are connected to k_j cells, and M_j is the number of cells in each group G_j ; $j = 1, 2, 3, \dots, J$. $N_0^{k_j}(t)$ denotes the number of cells of G_j in state 0, and $N_1^{k_j}(t)$ denotes the number of cells of G_j in state 1 at time t . Two variables satisfy the condition $N_0^{k_j}(t) + N_1^{k_j}(t) = M_j$ and the density function is

$$p(t + 1) = \sum_{j=1}^J p_j(t + 1) = \sum_{j=1}^J 1/N [N_{0 \rightarrow 1}^{k_j}(t) + N_{1 \rightarrow 1}^{k_j}(t)], \tag{11}$$

where $p_j(t + 1)$ is the probability that a cell in group G_j is in state 1 at time $t + 1$. Similar to that above, the final formulation is

$$p(t + 1) = \sum_{j=1}^J \frac{M_j}{N} R_1(t) [1 - p(t)]^{k_j - 1}, \tag{12}$$

where $R_1(t) \equiv \frac{N_0^{k_j}}{M_j} k_j p(t) + \frac{N_1^{k_j}}{M_j} [1 - p(t)]$ and the formulation is based on the evolution rule in Eq. (1), or

$$p(t+1) = \sum_{j=1}^J \frac{M_j}{N} R_2(t) p^{m-1}(t) [1 - p(t)]^{k_j - m}, \quad (13)$$

where $R_2(t) \equiv \frac{N_0^{k_j}}{M_j} \binom{k_j}{m} p(t) + \frac{N_1^{k_j}}{M_j} \binom{k_j}{m-1} [1 - p(t)]$, which is based on the generalized rule in Eq. (7) with $m \leq k_j$. Actually, m could be replaced by m_j in Eq. (13) and the new formulation indicates that each group G_j obeys its own evolution rule (with its own code number m_j).

The above idea can be extended to other totalistic rules. For instance, Rule 150 is $f[1,1,1] = f[0,0,1] = f[0,1,0] = f[1,0,0] = 1$, and $f = 0$ in the remaining four possible situations. The evolution rule must be rewritten as

$$c_n(t+1) = \begin{cases} 1 & c_n(t) + S_{nk}(t) = 1 \text{ or } k + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

and the density map of Rule 150 becomes

$$p(t+1) = p(t) \left\{ (1+k)[1-p(t)]^k + p^k(t) \right\}. \quad (15)$$

The bifurcation diagram of Eq. (15) is similar to Fig. 1c, and the similarity becomes stronger as k increases. The density maps of other totalistic rules can be constructed successfully.

4. Conclusion

In brief, a density map was formulated to describe the dynamical behavior of DCA based on Rule 22. Excellent agreement exists between the model and the original system, and the dynamics was studied using a bifurcation diagram and the largest Lyapunov exponent. The generalized Rule 22 was investigated, and the idea extended to other totalistic rules, such as Rules 104 and 150. The inhomogeneous situation is also discussed.

Synchronization behaviors in the coupled networks must be studied because such phenomena are exhibited in many coupled biological, physical, and even social systems. Due to the discreteness of DCA, the customary deterministic coupling applied in maps cannot be exercised. Accordingly, stochastic coupling technique is introduced to perform the interaction between two networks [20]. The annealed model (AM) allows for analytical calculations in two coupled identical DCA. However, when the numbers of connections k of two automata differ or more than two automata are coupled, the AM is useless. To solve the problem, ideas of stochastic coupling technique and the density map are combined, and deterministic coupled polynomial maps adopted to describe the density evolution of the coupled networks [23,24]. Our further works are that: modulating the coupling techniques and formulating the density evolution using a locally coupled map lattice, a globally coupled map lattice, or a power-law coupled map lattice. The coupled inhomogeneous automata are also under investigations.

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References

- [1] Wolfram S. Statistical mechanics of cellular automata. *Rev Mod Phys* 1983;55:601–44.
- [2] Wolfram S. Cellular automata and complexity. New York: Addison-Wesley; 1994.
- [3] Wolfram S. A new kind of science. Champaign: Wolfram Media; 2002.
- [4] Svozil K. Computational universes. *Chaos, Solitons & Fractals* 2005;25:845–59.
- [5] Toffoli T, Margolus N. Cellular automata machines: a new environment for modeling. Cambridge (MA): MIT; 1987.
- [6] Chopard B, Droz M. Cellular automata modeling of physical systems. Cambridge: Cambridge University Press; 1998.
- [7] Vichniac G. Simulating physics with cellular automata. *Physica D* 1984;10:96–115.
- [8] Chopard B, Luthi P, Droz M. Reaction–diffusion cellular automata model for the formation of Leisegang patterns. *Phys Rev Lett* 1994;72:1384–7.

- [9] Adamatzky A, Wuensche A, De Lacy Costello B. Glider-based computing in reaction–diffusion hexagonal cellular automata. *Chaos, Solitons & Fractals* 2006;27:287–95.
- [10] Gunstensen AK, Rothman DH, Zaleski S, Zanetti G. Lattice Boltzmann model of immiscible fluids. *Phys Rev A* 1991;43:4320–7.
- [11] Gravner J, Griffeath D. Cellular automaton growth on \mathbb{Z}^2 : Theorems, examples, and problems. *Adv Appl Math* 1998;21:241–304.
- [12] Chidyagwai P, Reiter CA. A local cellular model for growth on quasicrystals. *Chaos, Solitons & Fractals* 2005;24:803–12.
- [13] Wang B-H, Wang L, Hui P-M, Hu B. Cellular automaton model for one dimensional traffic flow with gradual acceleration and stochastic delay: Analytical approach. *Int J Nonlinear Sci Numer Simul* 2000;1(4):255–66.
- [14] Mao D, Wang B-H, Wang L, Hui P-K, Hu C-K. Chaotic traffic flow CA model in which only the cars following the trail of the ahead car can be delayed. *Int J Nonlinear Sci Numer Simul* 2003;4(3):239–50.
- [15] Kauffman SA. Metabolic stability and epigenesis in randomly constructed genetic nets. *J Theor Bio* 1969;22:437–67.
- [16] Kauffman SA. *The origins of order*. Cambridge: Oxford University Press; 1993.
- [17] Derrida B, Pomeau Y. Random networks of automata: a simple annealed approximation. *Europhys Lett* 1986;1:45–9.
- [18] Derrida B, Weisbuch G. Evolution of overlaps between configurations in random Boolean networks. *J Phys* 1986;47:1297–303.
- [19] Flyvbjerg H. An order parameter for networks of automata. *J Phys A* 1988;21:L955–60.
- [20] Morelli LG, Zanette DH. Synchronization of Kauffman networks. *Phys Rev E* 2001;63:036204.
- [21] Andrecut M, Ali MK. Chaos in a simple Boolean network. *Int J Mod Phys B* 2001;15:17–23.
- [22] Matache MT, Heidel J. Random Boolean network model exhibiting deterministic chaos. *Phys Rev E* 2004;69:056214.
- [23] Ho MC, Hung YC, Jiang IM. Stochastic coupling of two random Boolean networks. *Phys Lett A* 2005;344:36–42.
- [24] Hung YC, Ho MC, Lih JS, Jiang IM. Chaos synchronization of two stochastically coupled random Boolean networks. *Phys Lett A* 2006;356:35–43.
- [25] El Naschie MS, Rössler OE, Prigogine I, editors. *Quantum mechanics, diffusion and chaotic fractals*. Oxford: Pergamon Press; 1995. p. 129–50.