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Phase and anti-phase synchronization of two chaotic systems by using active control

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Abstract

Using techniques from active control theory, we demonstrate that two coupled chaotic systems can be phase and anti-phase synchronized. The techniques are applied to Lorenz, Rossler, and Chen systems. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Sensitivity to initial conditions is a generic feature of chaotic dynamical systems. Two chaotic systems starting from slightly different initial points in the state space separate away from each other with time. Therefore, how to control two chaotic systems to be synchronized has aroused a great deal of interest [1–6].

The concept of synchronization can be extended, such as generalized synchronization [6,7], phase synchronization [8], lag synchronization [9], and even anti-phase synchronization [10]. Recently, active control has been applied to synchronize two identical chaotic systems. Moreover, it is examined in different types of chaotic systems [11–13]. In this Letter, we generalize active control to phase and anti-phase synchronization, and simulate the method by using Lorenz, Rossler, and Chen systems.

2. Phase synchronization

First, we take two identical Lorenz systems into consideration:

$$\begin{cases} \dot{x}_1 = \sigma(y_1 - x_1), \\ \dot{y}_1 = rx_1 - y_1 - x_1 z_1, \\ \dot{z}_1 = x_1 y_1 - b z_1 \end{cases} \tag{1}$$

and

$$\begin{cases} \dot{x}_2 = \sigma(y_2 - x_2) + u_a(t), \\ \dot{y}_2 = rx_2 - y_2 - x_2 z_2 + u_b(t), \\ \dot{z}_2 = x_2 y_2 - b z_2 + u_c(t). \end{cases} \tag{2}$$

Using active control, we assume that the system with subscript 1 is to control the system with the subscript 2. There are three control functions $u_a(t)$, $u_b(t)$, and $u_c(t)$ to be determined.

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In order to ascertain the control functions, we can subtract (1) from (2), and using

$$x_3 = x_2 - x_1, \qquad y_3 = y_2 - y_1, \qquad z_3 = z_2 - z_1, \tag{3}$$

we can get

$$\begin{cases} \dot{x}_3 = \sigma(y_3 - x_3) + u_a(t), \\ \dot{y}_3 = rx_3 - y_3 - x_2z_2 + x_1z_1 + u_b(t), \\ \dot{z}_3 = x_2y_2 - x_1y_1 - bz_3 + u_c(t). \end{cases} \tag{4}$$

Then, defining the control functions as

$$\begin{cases} u_a(t) = V_a(t), \\ u_b(t) = x_2z_2 - x_1z_1 + V_b(t), \\ u_c(t) = -x_2y_2 + x_1y_1 + V_c(t), \end{cases} \tag{5}$$

this lead to

$$\begin{cases} \dot{x}_3 = \sigma(y_3 - x_3) + V_a(t), \\ \dot{y}_3 = rx_3 - y_3 + V_b(t), \\ \dot{z}_3 = -bz_3 + V_c(t). \end{cases} \tag{6}$$

According to the original method of active control, (6) can be rewritten as

$$\begin{pmatrix} V_a(t) \\ V_b(t) \\ V_c(t) \end{pmatrix} = A \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \tag{7}$$

and matrix A is given by

$$A = \begin{pmatrix} \sigma - 1 & -\sigma & 0 \\ -r & 0 & 0 \\ 0 & 0 & b - 1 \end{pmatrix}. \tag{8}$$

The three eigenvalues of the closed loop system are chosen as -1, -1, and -1. These choices will result in a stable system and the synchronization of two identical Lorenz systems [12,13].

As we know, the eigenvalues have much to do with the stability of a system. If we let m, n, and k be the eigenvalues, (6) and (7) can be rewritten in easier forms

$$\begin{cases} \dot{x}_3 = f_1(x_3, y_3, z_3) + V_a(t), \\ \dot{y}_3 = f_2(x_3, y_3, z_3) + V_b(t), \\ \dot{z}_3 = f_3(x_3, y_3, z_3) + V_c(t) \end{cases} \tag{9}$$

and

$$\begin{cases} V_a(t) = -f_1(x_3, y_3, z_3) + mx_3, \\ V_b(t) = -f_2(x_3, y_3, z_3) + ny_3, \\ V_c(t) = -f_3(x_3, y_3, z_3) + kz_3, \end{cases} \tag{10}$$

where $f_1(x_3, y_3, z_3)$, $f_2(x_3, y_3, z_3)$, and $f_3(x_3, y_3, z_3)$ are linear functions.

Based on the stability analysis, when all eigenvalues are smaller than zero, the system will be convergence. If there is any one of eigenvalues larger than zero, the system will be divergence. However, what may happen as any one (or more than one) of eigenvalues equals to zero while the others are less than zero? The answer is that phase synchronization will be achieved.

Now we choose m, n, and k to be equal to zero in turns, and (10) become

$$\begin{cases} V_a(t) = -\sigma(y_3 - x_3) + m_i x_3, \\ V_b(t) = -rx_3 + y_3 + n_i y_3, \\ V_c(t) = bz_3 + k_i z_3. \end{cases} \tag{11}$$

Making $(m_1, n_1, k_1) = (0, -1, -1)$, $(m_2, n_2, k_2) = (-1, 0, -1)$, $(m_3, n_3, k_3) = (-1, -1, 0)$, and the control functions $u_a(t)$, $u_b(t)$, $u_c(t)$ can be determined.

Numerical results

RK4 method is used to all of our simulations with time step being equal to 0.001. We select the parameters of two Lorenz systems as $\sigma = 10$, $r = 28$, $b = 2.66$ to ensure the chaotic behavior. The initial values are $x_1(0) = 0.5$, $y_1(0) = 1.0$, $z_1(0) = 1.5$ and $x_2(0) = 10.0$, $y_2(0) = 2.0$, $z_2(0) = 2.0$. And control inputs start at $t = 15$.

If (m, n, k) are selected as $(m_1, n_1, k_1) = (0, -1, -1)$, we can find signals x_1 and x_2 have the same shape while $t > 15$. To show x_1 and x_2 are phase synchronization, we sketch signal $x_3(t)$ $(= x_2 - x_1)$. As shown in Fig. 1, it is clear that x_3 is a constant after $t > 15$. Nevertheless, for y and z signals, they still are complete synchronization.

A general approach has been introduced by Gabor to define the phase of a time series, and it is based on the Hilbert transform. This definition is also very attractive in characterization of chaos [15]. In a more explicit form, the Hilbert transform of a time series $X(t)$ follows

$$\hat{X}(t) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{X(\tau)}{t - \tau} \, d\tau, \tag{12}$$

where P.V. means the Cauchy principal value for the integral. Thus, a new complex quantity $\Psi(t)$ can be introduced, i.e.,

$$\Psi(t) = X(t) + i\hat{X}(t) = A(t)e^{i\theta(t)}, \tag{13}$$

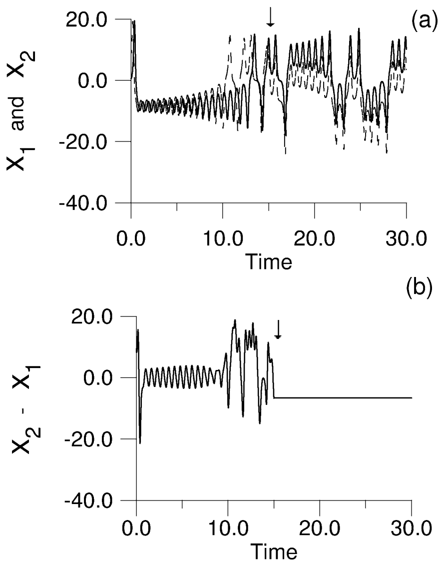

Fig. 1. The diagrams of two Lorenz systems become phase synchronization by using active control. (a) shows the time series of signals x_1 and x_2, and (b) shows the signal $x_3(t)$ $(= x_2 - x_1)$. The arrows indicate the time we begin to control.

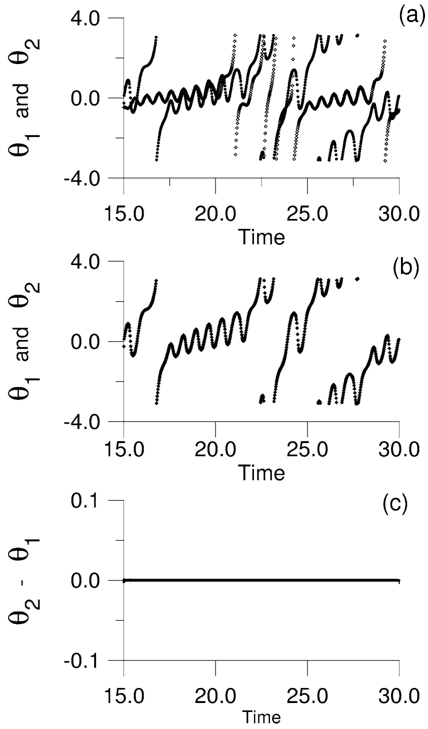

Fig. 2. (a) The phase of the two Lorenz signals x_1 and x_2 without being controlled after $t > 15$: (circles) θ_1, (diamonds) θ_2. (b) The phase of the two signals x_1 and x_2 being controlled after $t > 15$: (circles) θ_1, (diamonds) θ_2. (c) The phase different of θ_1 and θ_2 in (b).

where $\theta(t)$ is the phase and $A(t)$ is the amplitude [14,15] and they form a conjugate pair. When we use "Hilbert transform" to define the phase of the signals x_1 and x_2, the phase synchronization can be shown more clearly, as presented in Fig. 2.

Similarly, if (m_2, n_2, k_2) or (m_3, n_3, k_3) are used, $y_3(t)$ $(= y_2 - y_1)$ or $z_3(t)$ $(= z_2 - z_1)$ will be a constant, too (Fig. 3). Once more, when using (m_2, n_2, k_2), x or z signals are still complete synchronization; when using (m_3, n_3, k_3), x or y signals are still complete synchronization.

There are two things noticeable. First, in the condition of using (m_1, n_1, k_1), (m_2, n_2, k_2), or (m_3, n_3, k_3), we can also make non-zero numbers less than -1. If the eigenvalues get smaller, the convergence will become better. Second, we can also make (m, n, k) equal to $(0, 0, 0)$. After doing so, all dimensions become phase synchronization but not complete synchronization. However, the result of convergence is not as good as we got before.

In order to prove our theories, we use standard deviation ($\mathrm{SD} = \sqrt{(x(t) - \langle x(t)\rangle)^2/N}$) to test the results of simulation. We make m equal to zero, and then change n and k from -1 to -100 simultaneously to analyze signal $x_3(t)$ $(= x_2 - x_1)$ after $t > 15$. As shown in Fig. 4, when n and k get smaller, the values of SD also become smaller. That is to say, the convergence becomes better.

Let us outline another tool—mutual information [16,17]. With two time series (labeled by the subscripts i and j), one can divide the output range into S intervals (S states) and the probability P_l which is the chance for a l state to appear can be deduced. Thus, one can deduce the Shannon entropy

$$H_i = -\sum_{l=0}^{S-1} P_l \ln P_l,$$

for time series i, and the mutual information between two time series

$$M_{i,j} = H_i + H_j - H_{i,j},$$

where $H_{i,j} = -\sum_{l,m} P_{l,m} \ln P_{l,m}$ is the joint Shannon entropy in which the joint probability $P_{l,m}$ is the chance such that a l state occurs to time series i,

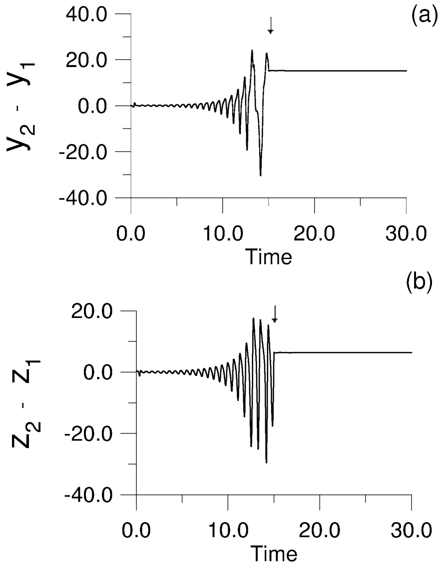

Fig. 3. (a) $y_3(t)$ $(= y_2 - y_1)$ when (m_2, n_2, k_2) is used. (b) $z_3(t)$ $(= z_2 - z_1)$ when (m_3, n_3, k_3) is used.

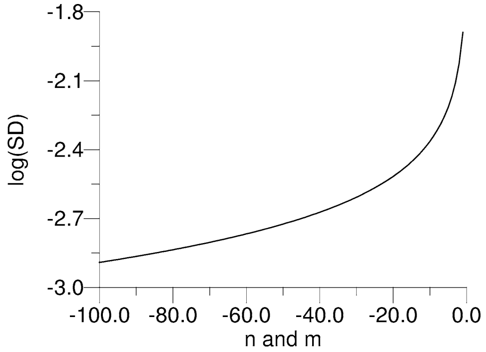

Fig. 4. The standard deviation (SD) of the signal $x_3(t)$ $(= x_2 - x_1)$, where m equals to zero, and n and k change from -1 to -100 simultaneously.

while a m state appears for time series j. Herein, we label the mutual information between the variables $X_i(t)$ and $Y_i(t)$ as $M(X_i, Y_i)$. To simplify the notation, $M(x_1, x_2) \equiv M_x$, $M(y_1, y_2) \equiv M_y$, and $M(z_1, z_2) \equiv M_z$. For M_x, we choose $n = k = -1$ and change m from 0 to -2 to observe the variance of M_x. Similarly, for M_y we make $m = k = -1$ with changing n from 0 to -2, and for M_z let $m = n = -1$ with changing k from 0 to -2. We control two systems after $t = 10$, and analyze MI after $t > 15$. In the range $-0.1 <$ eigenvalues $\leqslant 0$, the value of MI decreases, and it implies the loss of phase synchronization. When

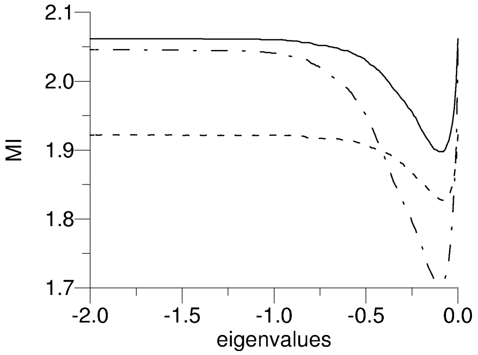

Fig. 5. The diagram of different mutual information versus their responding eigenvalues. The mutual information M_x is labeled as a solid line, mutual information M_y is labeled as a dashed line, and M_z is labeled as a dash-dotted line.

eigenvalues are less than -0.1, M_x, M_y, and M_z will increase instead. And it means the gain of complete synchronization. The outcomes are presented in Fig. 5.

Fig. 5 presents another interesting result. For M_z, in the range $-0.8 <$ eigenvalues $\leqslant 0$, the rate of gaining or losing information is larger than the rate of M_x or M_y. That is to say, it is easier for z signal to become phase synchronization or lose phase synchronization than x signal or y signal dose. We are interested in this phenomenon, and would like to investigate it further.

The methods are also used to control Rossler system,

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c), \end{cases} \tag{14}$$

and Chen system,

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz. \end{cases} \tag{15}$$

Similar results can be got in Figs. 6 and 7. In other words, phase synchronization is successfully made.

3. Anti-phase synchronization

Then, we are going to control two identity systems to anti-phase synchronization.

Making use of the same systems applied in (1) and (2), we add (1) to (2) instead of subtracting (1)

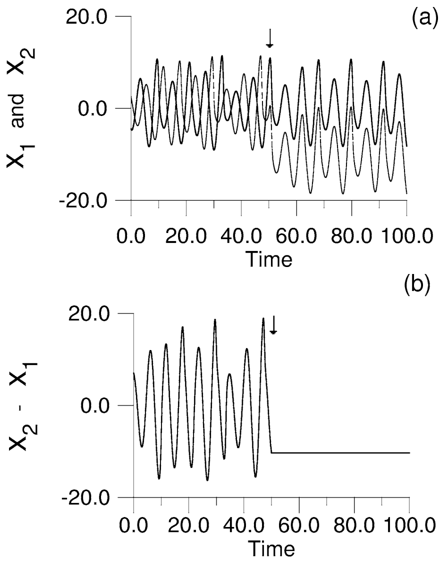

Fig. 6. The diagrams of two Rossler systems become phase synchronization by using active control, where $a = 0.2$, $b = 0.5$, $c = 5.7$. (a) shows the time series of signals x_1 and x_2, and (b) shows the signal $x_3(t)$ $(= x_2 - x_1)$. The arrows indicate the time we begin to control.

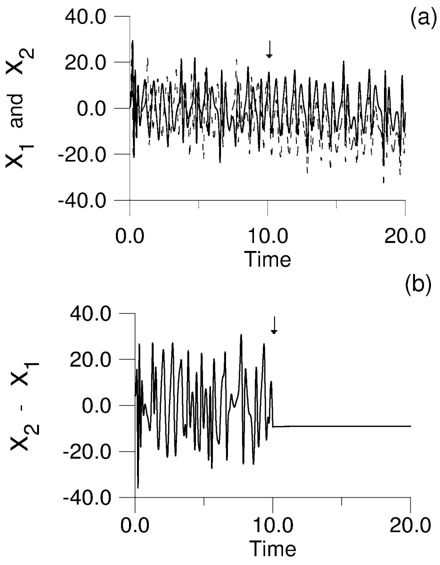

Fig. 7. The diagrams of two Chen systems become phase synchronization by using active control, where $a = 35$, $b = 3$, $c = 38$. (a) shows the time series of signals x_1 and x_2, and (b) shows the signal $x_3(t)$ $(= x_2 - x_1)$. The arrows indicate the time we begin to control.

from (2). Using

$$x_4 = x_2 + x_1, \qquad y_4 = y_2 + y_1, \qquad z_4 = z_2 + z_1 \tag{16}$$

and following the same techniques, we can get

$$\begin{cases} \dot{x}_4 = \sigma(y_4 - x_4) + u_a(t), \\ \dot{y}_4 = rx_4 - y_4 - x_2z_2 - x_1z_1 + u_b(t), \\ \dot{z}_4 = x_2y_2 + x_1y_1 - bz_4 + u_c(t) \end{cases} \tag{17}$$

and

$$\begin{cases} u_a(t) = V_a(t), \\ u_b(t) = x_2z_2 + x_1z_1 + V_b(t), \\ u_c(t) = -x_2y_2 - x_1y_1 + V_c(t). \end{cases} \tag{18}$$

From (9) and (10), we can decide $v_a(t)$, $v_b(t)$, and $v_c(t)$ quickly:

$$\begin{cases} V_a(t) = -\sigma(y_4 - x_4) + mx_4, \\ V_b(t) = -rx_4 + y_4 + ny_4, \\ V_c(t) = bz_4 + kz_4. \end{cases} \tag{19}$$

When making all eigenvalues m, n, and k are equal to -1 (of course, other values which are smaller than zero can be chosen), (17) will converge. If the eigenvalues get smaller, the convergence will become better. In other words, the values $x_1 + x_2$, $y_1 + y_2$, and $z_1 + z_2$ will converge to zero and anti-phase synchronization is reached.

Numerical results

Once more, we select the parameters of two Lorenz systems as $\sigma = 10$, $r = 28$, $b = 2.66$ to ensure the chaotic behavior. The initial values are $x_1(0) = 0.5$, $y_1(0) = 1$, $z_1(0) = 1.5$ and $x_2(0) = 0.6$, $y_2(0) = 0.9$, $z_2(0) = 1.1$. And control inputs start at $t = 15$.

Fig. 8 shows the anti-phase synchronization of two Lorenz systems: (a) displays x_1 and x_2 signals, and (b) displays the $x_4(t)$ $(= x_2 + x_1)$ signal. We can get similar results from y and z signals.

4. Conclusion

By means of techniques from active control theory, we have easily controlled chaotic systems to phase and anti-phase synchronization. We believe that the techniques can still be generalized.

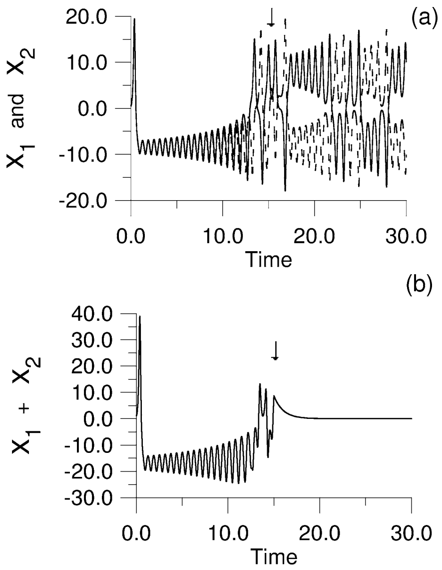

Fig. 8. The diagrams of two Lorenz systems become anti-phase synchronization by using active control. (a) shows the time series of signals x_1 and x_2, and (b) shows the signal $x_4(t)$ $(= x_2 + x_1)$. The arrows indicate the time we begin to control.

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