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Reduced-order synchronization of chaotic systems with parameters unknown

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Abstract

In this Letter, we investigate the reduced-order synchronization of uncertain chaotic systems. Based upon the parameters modulation and the adaptive control techniques, we control the response system to be the drive system even though their orders are different and their parameters are unknown. The techniques are successfully applied to two examples: generalized Lorenz system (fourth order) and Lü system (third order); Rössler hyperchaotic system (fourth order) and Rössler system (third order). Furthermore, the effect of control modulator k is under our discussions.

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Keywords: Synchronization

1. Introduction

A typical feature of the chaotic system is its extreme sensitivity to initial conditions. Slight errors occurring in initial states of two identical oscillators will lead to completely different trajectories after enough transient time. Therefore, how to control chaotic systems has been an important topic in nonlinear science. Generally speaking, chaos suppression and chaos synchronization are the two different concepts about control. Chaos suppression mainly consists in the stabilization of the chaotic system around periodic orbits or fixed points. The OGY method is the representative research on chaos suppression [1]. In 1990, Ott, Grebogi, and Yorke introduced a linear feedback method to stabilize unstable periodic orbits in chaotic systems. Because the method does not require prior

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knowledge of the governing equations, it generates widespread interest and has been applied to optical experiment and communication extensively [2–6].

Methods for synchronizing chaotic systems developed simultaneously with the studies of chaos suppression. Since the study made by Pecora and Carroll [7], the investigation of chaotic synchronization has attracted a lot of attention owing to various potential applications, such as secure communication [8,9], neuron systems [10], and the study of laser dynamics [11,12]. In general, synchronization is understood as the adjustment of the states of coupled systems. The trajectories of different subsystems become identical in a strong-coupling condition; the subsystems may be less correlated in a weaker-coupling condition. Depending on different coupling methods and coupling strengths, several different types of synchronization can be observed: complete synchronization (CS), phase synchronization (PS) [13], lag synchronization (LS) [14], anticipation synchronization [15], generalized synchronization (GS) [16], and so on. They represent the different degrees of correlation in interacting systems.

This Letter addresses the chaotic synchronization based upon nonlinear controllers. The condition, when two chaotic systems are identical, has been extremely investigated [7,17]. However, the synchronization of chaotic systems with different orders is far less understood [18–21]. Because the order of the slave oscillator is lower than that of the master system, the synchronization is only attained in reduced order. Such a problem is pertinent in the study of neural networks [22–24]. For instance, the output from higher-order neurons always drives the neurons with lower-order in the subsystem. Similar phenomena can be expected in the human cardiovascular system [25,26]. Moreover, studying such problems can help us elucidate the coherent behavior of complex systems; notwithstanding the inherent interest in the problem itself.

In this Letter, we take the drive-response synchronization into consideration. That is, the second system is driven by the first one but the behavior of the first system is not affected. The two systems are called the master and the slave system individually. By using the adaptive control and the parameters modulation techniques [27–33], we control the slave system to be the master successfully even though their orders are different and their parameters are unknown. Two examples are demonstrated: generalized Lorenz system (fourth order) and Lü system (third order); Rössler hyperchaotic system (fourth order) and Rössler system (third order). Furthermore, the effect of control modulator k is under our discussions.

The rest of this Letter is organized as follows. Section 2 presents the strategies of adaptive control and the parameters identification techniques. Sections 3 and 4 use generalized Lorenz system and Lü system, Rössler hyperchaotic system and Rössler system to perform synchronization respectively. Moreover, we discuss the effect of control modulator k. Conclusions and further works are finally drawn in Section 5.

2. Problem formulation

Consider the following system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\mathbf{P}, \tag{1}$$

where $\mathbf{x} \in R^m$ is the state vector of the system, $\mathbf{f}: R^m \to R^m$ is a continuous vector function including nonlinear terms, $\mathbf{F}: R^m \to R^{m\times k}$, and $\mathbf{P} \in R^k$ is the vector of system parameters. Eq. (1) is considered as the master system. The systems studied in this Letter depend linearly on the parameters, and many well-known chaotic (hyperchaotic) systems belong to Eq. (1). Similarly, the slave system is given by

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) + \mathbf{G}(\mathbf{y})\boldsymbol{\Theta} + \mathbf{U}, \tag{2}$$

where $\mathbf{y} \in R^n$ is the state vector, $\mathbf{g}: R^n \to R^n$ is a continuous vector function, $\mathbf{G}: R^n \to R^{n\times l}$, and $\boldsymbol{\Theta} \in R^l$ is the parameter vector. The purpose of chaos synchronization is to design a controller $\mathbf{U}$ ($\mathbf{U} \in R^n$), which is able to synchronize the state of the master system and the slave system.

When order $n = m$, $k = l$ and the functions $\mathbf{f} = \mathbf{g}$, $\mathbf{F} = \mathbf{G}$, the slave system is identical to the master system, and the CS problem has been well studied. When two systems satisfy the condition $n < m$ (of course $\mathbf{f} \neq \mathbf{g}$, and

$\mathbf{F} \neq \mathbf{G}$), that is, the order of the slave oscillator is lower than that of the master system, the synchronization is only attained in reduced order. Actually, reduced-order synchronization is the problem of controlling a slave system to be the projection of the master system. Therefore, we can divide the master system into two parts.

The projection:

$$\dot{\mathbf{x}}_p = \mathbf{f}_p(\mathbf{x}) + \mathbf{F}_p(\mathbf{x})\mathbf{P}, \tag{3}$$

where $\mathbf{x}_p \in R^n$, $\mathbf{f}_p : R^m \to R^n$, and $\mathbf{F}_p : R^m \to R^{n\times k}$.

The rest:

$$\dot{\mathbf{x}}_r = \mathbf{f}_r(\mathbf{x}) + \mathbf{F}_r(\mathbf{x})\mathbf{P}, \tag{4}$$

where $\mathbf{x}_r \in R^u$, $\mathbf{f}_r : R^m \to R^u$, $\mathbf{F}_r : R^m \to R^{u\times k}$ and order n, u satisfy $n + u = m$. With a suitable controller, the reduced-order synchronization between two different systems can be achieved, i.e.,

$$\lim_{t\to+\infty} \|\mathbf{y} - \mathbf{x}_p\| = 0. \tag{5}$$

Defining the error vector $\mathbf{e} = \mathbf{y} - \mathbf{x}_p$ ($\mathbf{e} \in R^n$), we subtract (3) from (2) and get

$$\dot{\mathbf{e}} = \mathbf{g}(\mathbf{y}) + \mathbf{G}(\mathbf{y})\boldsymbol{\Theta} + \mathbf{U} - \mathbf{f}_p(\mathbf{x}) - \mathbf{F}_p(\mathbf{x})\mathbf{P} = \mathbf{h}(\mathbf{e},\mathbf{x}) + \mathbf{G}(\mathbf{e},\mathbf{x})\boldsymbol{\Theta} - \mathbf{F}_p(\mathbf{x})\mathbf{P} + \mathbf{U}, \tag{6}$$

where $\mathbf{h} : R^n \times R^m \to R^n$ is a continuous vector function. In practical situations, the parameters belonging to the master and the slave system are always unknown. Therefore, by using the adaptive control and the parameters identification techniques [29,30], the controller can be decided as

$$\mathbf{U} = \mathbf{H}(\mathbf{e},\mathbf{x}) - \mathbf{G}(\mathbf{e},\mathbf{x})\hat{\boldsymbol{\Theta}} + \mathbf{F}_p(\mathbf{x})\hat{\mathbf{P}}, \tag{7}$$

where $\mathbf{H} : R^n \times R^m \to R^n$, $\hat{\boldsymbol{\Theta}}$ and $\hat{\mathbf{P}}$ are the estimated vector of unknown parameters, and the updating laws of the estimated parameters are given by

$$\begin{cases} \dot{\hat{\boldsymbol{\Theta}}} = \mathbf{G}^T(\mathbf{y})\mathbf{e}^T = \mathbf{G}^T(\mathbf{e},\mathbf{x})\mathbf{e}^T, \\ \dot{\hat{\mathbf{P}}} = -\mathbf{F}_p^T(\mathbf{x})\mathbf{e}^T. \end{cases} \tag{8}$$

Assume a positive Lyapunov function $V = \frac{1}{2}(\mathbf{e}^T\mathbf{e} + \tilde{\boldsymbol{\Theta}}^T\tilde{\boldsymbol{\Theta}} + \tilde{\mathbf{P}}^T\tilde{\mathbf{P}})$ (where $\tilde{\mathbf{P}} = \hat{\mathbf{P}} - \mathbf{P}$ and $\tilde{\boldsymbol{\Theta}} = \hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}$). With the choice of the updating laws above and reasonable control function $\mathbf{H}(\mathbf{e},\mathbf{x})$, the time rate of change of V along the solution in Eq. (6) will be smaller than zero. In other words, the error vector will approach to zero as time goes on and the states of the slave system and projected master system are synchronized asymptotically. Noteworthily, the design of $\mathbf{H}(\mathbf{e},\mathbf{x})$ sensitively depends on the considered dynamical system. To simplify the question, the details of $\frac{dV}{dt} < 0$ will be shown in the following sections via generalized Lorenz system and Lü system, hyperchaotic Rössler system and Rössler system individually.

3. Synchronization of generalized Lorenz system and Lü system

In this section, we take generalized Lorenz system and Lü system into consideration. Recently, by including the effects of external rotation, Stenflo showed that low-frequency, short-wavelength acoustic gravity waves can be described by a system of four coupled nonlinear ordinary differential equations [34]. The system is similar with Lorenz system but with a new control parameter c_1 and a new variable x_4. It is called generalized Lorenz system and given by:

$$\dot{x}_1 = a_1(x_2 - x_1) + c_1x_4,$$

$$\dot{x}_2 = r_1x_1 - x_1x_3 - x_2,$$
$$\dot{x}_3 = x_1x_2 - b_1x_3,$$
$$\dot{x}_4 = -x_1 - a_1x_4, \tag{9}$$

where c $(c > 0)$ is the rotation number. The nonlinear differential equations that describe Lü system are:

$$\dot{y}_1 = a_2(y_2 - y_1),$$
$$\dot{y}_2 = -y_1y_3 + c_2y_2,$$
$$\dot{y}_3 = y_1y_2 - b_2y_3, \tag{10}$$

where a_2, b_2, c_2 are three positive parameters.

Our purpose is to synchronize Lü system with projection of generalized Lorenz system. Therefore, the master system is the projection part of Eq. (9), and it can be presented in the form of:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}0\\ -x_1x_3 - x_2\\ x_1x_2\end{pmatrix} + \begin{pmatrix}x_2 - x_1 & 0 & 0 & x_4\\ 0 & x_1 & 0 & 0\\ 0 & 0 & -x_3 & 0\end{pmatrix}\begin{pmatrix}a_1\\ r_1\\ b_1\\ c_1\end{pmatrix}. \tag{11}$$

Similarly, the slave system becomes

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix} = \begin{pmatrix}0\\ -y_1y_3\\ y_1y_2\end{pmatrix} + \begin{pmatrix}y_2 - y_1 & 0 & 0\\ 0 & y_2 & 0\\ 0 & 0 & -y_3\end{pmatrix}\begin{pmatrix}a_2\\ c_2\\ b_2\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}, \tag{12}$$

where vector $(u_1, u_2, u_3)^T$ is the controller and all parameters a_1, r_1, b_1, c_1, a_2, c_2, b_2 are unknown.

Defining the error state $e_i = y_i - x_i$ $(i = 1, 2, 3)$, we subtract (11) from (12) and get

$$\begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{pmatrix} = \begin{pmatrix}0\\ -e_1e_3 - e_1x_3 - x_1e_3 + x_2\\ e_1e_2 + e_1x_2 + x_1e_2\end{pmatrix}$$
$$+ \begin{pmatrix}y_2 - y_1 & 0 & 0\\ 0 & y_2 & 0\\ 0 & 0 & -y_3\end{pmatrix}\begin{pmatrix}a_2\\ c_2\\ b_2\end{pmatrix} - \begin{pmatrix}x_2 - x_1 & 0 & 0 & x_4\\ 0 & x_1 & 0 & 0\\ 0 & 0 & -x_3 & 0\end{pmatrix}\begin{pmatrix}a_1\\ r_1\\ b_1\\ c_1\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}. \tag{13}$$

Based upon Eq. (7) and with the simplest choice of control function $\mathbf{H}(\mathbf{e}, \mathbf{x})$, we get the controller:

$$u_1 = -k_1e_1 - (e_2 - e_1)\hat{a}_2 + (x_1 - x_2)(\hat{a}_2 - \hat{a}_1) + x_4\hat{c}_1,$$
$$u_2 = -k_2e_2 + e_1x_3 - (1 + \hat{c}_2)x_2 - e_2\hat{c}_2 + x_1\hat{r}_1,$$
$$u_3 = -k_3e_3 - e_1x_2 + e_3\hat{b}_2 + x_3(\hat{b}_2 - \hat{b}_1), \tag{14}$$

where k_1, k_2, k_3 are the control modulators, which are positive constants and $\hat{a}_1$, $\hat{r}_1$, $\hat{b}_1$, $\hat{c}_1$, $\hat{a}_2$, $\hat{c}_2$, $\hat{b}_2$ are the estimated value of parameters, which obey the updating laws:

$$\dot{\hat{a}}_1 = -(x_2 - x_1)e_1,$$
$$\dot{\hat{r}}_1 = -x_1e_2,$$
$$\dot{\hat{b}}_1 = x_3e_3,$$
$$\dot{\hat{c}}_1 = -x_4e_1, \tag{15}$$

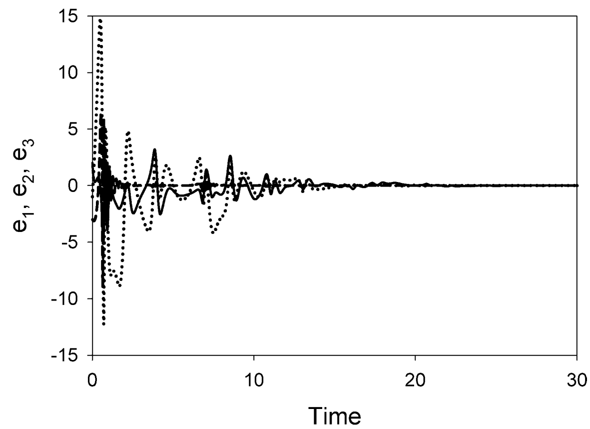

Fig. 1. The diagram presents the errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ between generalized Lorenz system and Lü system. $e_1(t) = y_1(t) - x_1(t)$ is labeled as a solid line, $e_2(t) = y_2(t) - x_2(t)$ is labeled as a dotted line and $e_3(t) = y_3(t) - x_3(t)$ is labeled as a dashed line.

and

$$\begin{aligned}
\dot{\hat{a}}_2 &= (y_2 - y_1)e_1 = e_2e_1 - e_1^2 + (x_2 - x_1)e_1, \\
\dot{\hat{c}}_2 &= y_2e_2 = e_2^2 + x_2e_2, \\
\dot{\hat{b}}_2 &= -y_3e_3 = -e_3^2 - x_3e_3.
\end{aligned} \tag{16}$$

It is necessary to show that the synchronization is realizable. Defining the errors between unknown and estimated parameters as $\tilde{a}_1 = \hat{a}_1 - a_1$, $\tilde{r}_1 = \hat{r}_1 - r_1$, $\tilde{b}_1 = \hat{b}_1 - b_1$, $\tilde{c}_1 = \hat{c}_1 - c_1$, $\tilde{a}_2 = \hat{a}_2 - a_2$, $\tilde{c}_2 = \hat{c}_2 - c_2$, $\tilde{b}_2 = \hat{b}_2 - b_2$, we can choose a Lyapunov function

$$V = \frac{1}{2}\left(\mathbf{e}^T\mathbf{e} + \tilde{a}_1^2 + \tilde{r}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{a}_2^2 + \tilde{c}_2^2 + \tilde{b}_2^2\right). \tag{17}$$

Taking the time derivation of V along the trajectories of the error dynamical system (13) leads to

$$\begin{aligned}
\frac{dV}{dt} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \tilde{a}_1\dot{\hat{a}}_1 + \tilde{r}_1\dot{\hat{r}}_1 + \tilde{b}_1\dot{\hat{b}}_1 + \tilde{c}_1\dot{\hat{c}}_1 + \tilde{a}_2\dot{\hat{a}}_2 + \tilde{c}_2\dot{\hat{c}}_2 + \tilde{b}_2\dot{\hat{b}}_2 \\
&= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 < 0,
\end{aligned} \tag{18}$$

where k_1, k_2, k_3 are arbitrarily chosen positive modulators. Since V is a positive definite function and $\dot{V}$ is a negative definite function, the error states $\lim_{t\to\infty} \|\mathbf{e}(t)\| = 0$. In other words, the states of controlled slave system and the projection part of master system are globally synchronized asymptotically.

RK4 method is used to our simulations with time step being equal to 0.0001. We select the parameters of the master system as $a_1 = 1.0$, $r_1 = 26.0$, $b_1 = 0.7$, $c_1 = 1.5$ to ensure the chaotic behavior and the parameters of the slave system as $a_2 = 36.0$, $c_2 = 20.0$, $b_2 = 3.0$. The initial values are $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = -1$, $x_4(0) = 1$ and $y_1(0) = -1$, $y_2(0) = 1$, $y_3(0) = 2$. The estimated parameters start from $\hat{a}_1(0) = 0.2$, $\hat{r}_1(0) = 10.0$, $\hat{b}_1 = 2.0$, $\hat{c}_1 = 5.0$, $\hat{a}_2 = 25.0$, $\hat{c}_2 = 30.0$, $\hat{b}_2 = 5.0$. We choose control modulators $k_1 = k_2 = k_3 = 1.0$. Fig. 1 shows the errors between two chaotic systems. When the errors approach to zero, the reduced-order synchronization of two uncertain chaotic systems is realized.

4. Synchronization of Rössler hyperchaotic system and Rössler system

Rössler hyperchaotic system with two positive Lyapunov exponents has been provided by Rössler [35] in the form of ordinary differential equations describing dynamics of some hypothetical chemical reaction

$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_3,\\
\dot{x}_2 &= x_1 + a_1 x_2 + x_4,\\
\dot{x}_3 &= b_1 + x_1 x_3,\\
\dot{x}_4 &= -c_1 x_3 + d_1 x_4,
\end{aligned} \tag{19}$$

where a_1, b_1, c_1, d_1 are dimensionless parameters. The nonlinear differential equations that describe Rössler system are:

$$\begin{aligned}
\dot{y}_1 &= -y_2 - y_3,\\
\dot{y}_2 &= y_1 + a_2 y_2,\\
\dot{y}_2 &= b_2 + y_3(y_1 - c_2).
\end{aligned} \tag{20}$$

Similarly, we regard the mater as the projection part of Rössler hyperchaotic system, which is presented by

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}-x_2 - x_3\\ x_1 + x_4\\ x_1 x_3\end{pmatrix} + \begin{pmatrix}0 & 0\\ x_2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_1\\ b_1\end{pmatrix}, \tag{21}$$

and the slave system is given by

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix} = \begin{pmatrix}-y_2 - y_3\\ y_1\\ y_1 y_3\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ y_2 & 0 & 0\\ 0 & 1 & -y_3\end{pmatrix}\begin{pmatrix}a_2\\ b_2\\ c_2\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}, \tag{22}$$

where $(u_1, u_2, u_3)^T$ is the controlling vector and parameters a_1, b_1, a_2, b_2, c_2 are uncertain.

Defining the error state $e_i = y_i - x_i$ $(i = 1, 2, 3)$, we subtract (21) from (22) and get

$$\begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{pmatrix} = \begin{pmatrix}-e_2 - e_3\\ e_1 - x_4\\ e_1 e_3 + e_1 x_3 + x_1 e_3\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ y_2 & 0 & 0\\ 0 & 1 & -y_3\end{pmatrix}\begin{pmatrix}a_2\\ b_2\\ c_2\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} - \begin{pmatrix}0 & 0\\ x_2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_1\\ b_1\end{pmatrix}. \tag{23}$$

The purpose of control is to decide a controller and the update law of estimated parameters so that the states of master system and slave system are synchronized asymptotically. Based upon Eq. (7) and with the suitable choice of control function $\mathbf{H}(\mathbf{e}, \mathbf{x})$, we can get the controller:

$$\begin{aligned}
u_1 &= -k_1 e_1 - e_3 x_3 - e_3^2,\\
u_2 &= -k_2 e_2 + x_4 - e_2 \hat{a}_2 + x_2(\hat{a}_1 - \hat{a}_2),\\
u_3 &= -k_3 e_3 + e_3(\hat{c}_2 - x_1) + e_1 + (\hat{b}_1 - \hat{b}_2) + x_3 \hat{c}_2.
\end{aligned} \tag{24}$$

Simultaneously, the updating laws of the estimated parameters are given by

$$\begin{aligned}
\dot{\hat{a}}_1 &= -x_2 e_2,\\
\dot{\hat{b}}_1 &= -e_3,
\end{aligned} \tag{25}$$

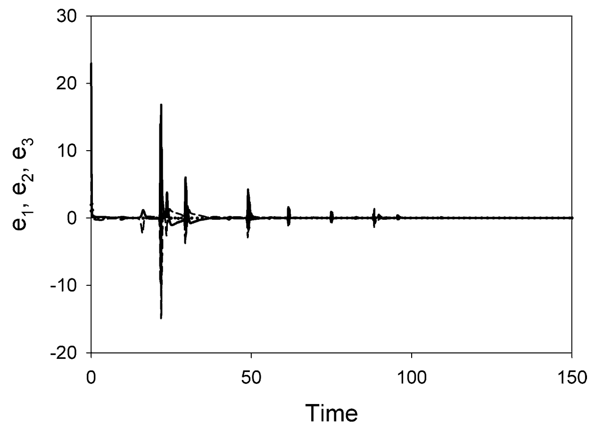

Fig. 2. The diagram presents the synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ of the unidirectional coupled Rössler hyperchaotic system and Rössler system. $e_1(t)$ is labeled as a solid line, $e_2(t)$ is labeled as a dotted line and $e_3(t)$ is labeled as a dashed line.

and

$$\begin{aligned} \dot{\hat{a}}_2 &= (e_2 + x_2)e_2, \\ \dot{\hat{b}}_2 &= e_3, \\ \dot{\hat{c}}_2 &= -(e_3 + x_3)e_3. \end{aligned} \tag{26}$$

Using Lyapunov stability theory, one can show that synchronization is attainable for all initial conditions. Let the errors between unknown and estimated parameters are $\tilde{a}_1 = \hat{a}_1 - a_1$, $\tilde{b}_1 = \hat{b}_1 - b_1$, $\tilde{a}_2 = \hat{a}_2 - a_2$, $\tilde{b}_2 = \hat{b}_2 - b_2$, $\tilde{c}_2 = \hat{c}_2 - c_2$, and then we can choose the positive Lyapunov function as:

$$V = \frac{1}{2}\left(\mathbf{e}^T\mathbf{e} + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2 + \tilde{c}_2^2\right). \tag{27}$$

Similarly, taking the time derivation of V along the trajectories in the error dynamical system (23), it is easy to show that:

$$\frac{dV}{dt} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 < 0, \tag{28}$$

where k_1, k_2, k_3 are positive modulators. Since $\dot{V}$ is a negative definite function, the error states $\lim_{t\to\infty}\|\mathbf{e}(t)\| = 0$. In another word, this choice will lead the errors to converge to zero as time passes, and hence the reduced-order synchronization is achieved.

The parameters of two systems are selected as the typical value, $a_1 = 0.25$, $b_1 = 3.00$, $c_1 = 0.50$, $d_1 = 0.05$, $a_2 = 0.20$, $b_2 = 0.20$, and $c_2 = 5.70$. The initial conditions are $x_1(0) = -20$, $x_2(0) = 0$, $x_3(0) = 0$, $x_4(0) = 15$ and $y_1(0) = -1$, $y_2(0) = 1$, $y_3(0) = 2$. The estimated parameters start from $\hat{a}_1(0) = 1.0$, $\hat{b}_1 = 5.0$, $\hat{a}_2 = 1.0$, $\hat{b}_2 = 2.0$, $\hat{c}_2 = 5.0$. We choose control modulators $k_1 = k_2 = k_3 = 2.0$ and time step $dt = 0.001$. The results of reduced-order synchronization are presented in Fig. 2.

Moreover, the reduced-order synchronization can be improved by increasing the three positive modulators k_1, k_2, and k_3. To quantify the synchronization, we evaluate mean-square error (MSE) between the slave system (y_1, y_2, y_3) and the master system (x_1, x_2, x_3) after controlling. The MSE is defined as

$$\mathrm{MSE} = \frac{1}{T}\int_0^T \left[\mathbf{x}(t) - \mathbf{y}(t)\right]^2 dt. \tag{29}$$

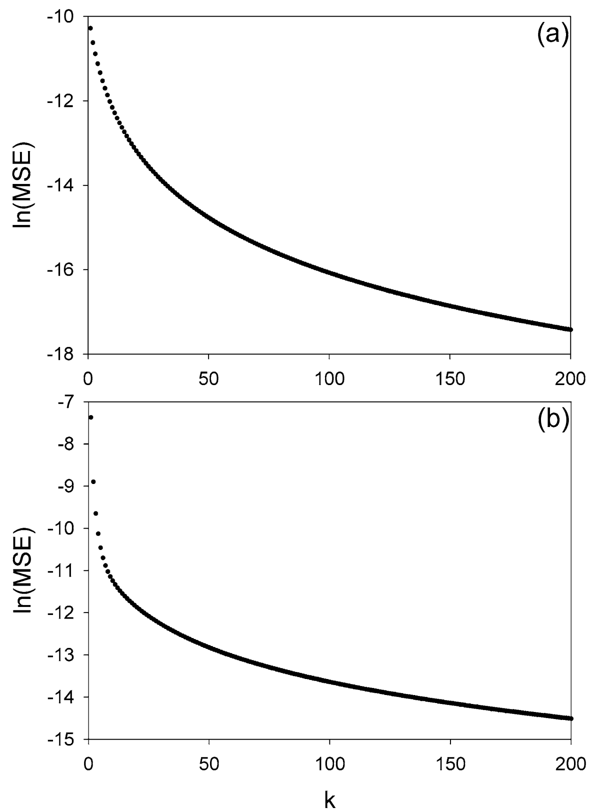

Fig. 3. The diagram presents the values of $\ln(\mathrm{MSE})$ versus the modulators k, where k vary from 1 to 200 with step 1. (a) illustrates the $\ln(\mathrm{MSE})$ of synchronized generalized Lorenz system and Lü system. (b) demonstrates the $\ln(\mathrm{MSE})$ of synchronized Rössler hyperchaotic system and Rössler system.

Consider the error signal in the first example, the value of MSE is calculated with various modulators. To simplify the analysis, we set $k_1 = k_2 = k_3 = k$, and change k from 1 to 200 in steps of 1. The transitional signals are eliminated, and T is set as large as possible. As Fig. 3(a) presents, MSE gets smaller with the increase of k, which implies that the reduced-order synchronization will be excellent if the modulators k are sufficiently large. Additionally, the values of MSE between Rössler hyperchaotic and Rössler systems demonstrate the similar property. Fig. 3(b) illustrates the results.

5. Conclusion

Based on Lyapunov stability theory, we propose the adaptive control and parameters modulation techniques to synchronize two chaotic systems with different order though their parameters are uncertain. The simulation results show that the sates of generalized Lorenz system and Lü system; hyperchaotic Rössler system and Rössler system are synchronized asymptotically. With the increase of modulators k, the reduced-order synchronization can be improved further.

Noteworthily, in our examples, the estimation value of parameters will not be identical to the real parameters. This is because the error systems and the parameter updating laws may possess multi-equilibrium points [36]. How to find the unique solution to the error systems and get the correct parameters are our further works.

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References

[1] E. Ott, C. Grebogi, J.A. Yorke, Phys. Rev. Lett. 64 (1990) 1196.
[2] U. Dressler, G. Nitsche, Phys. Rev. Lett. 68 (1992) 1.
[3] E.R. Hunt, Phys. Rev. Lett. 67 (1991) 1953.
[4] R. Roy, T.W. Murphy, T.D. Maier, Z. Gills, E.R. Hunt, Phys. Rev. Lett. 68 (1992) 1259.
[5] V. Petrov, V. Gáspár, J. Masere, K. Showalter, Nature (London) 361 (1993) 240.
[6] S. Hayes, C. Grebogi, E. Ott, A. Mark, Phys. Rev. Lett. 73 (1994) 1781.
[7] L. Pecora, T. Carroll, Phys. Rev. Lett. 64 (1990) 821.
[8] L. Kocarev, U. Parlitz, Phys. Rev. Lett. 74 (1995) 5028.
[9] U. Parlitz, L. Kocarev, T. Stojanovski, H. Preckel, Phys. Rev. E 53 (1996) 4351.
[10] J.W. Shuaim, K.W. Wong, Phys. Rev. E 57 (1998) 7002.
[11] R. Roy, K.S. Thornburg Jr., Phys. Rev. Lett. 72 (1994) 2009.
[12] D.Y. Tang, R. Dykstra, N.R. Heckenberg, Phys. Rev. A 54 (1996) 5317.
[13] M.G. Rosenblum, A.S. Pikovsky, J. Kurths, Phys. Rev. Lett. 76 (1996) 1804;
E. Rosa, E. Ott, M.H. Hess, Phys. Rev. Lett. 80 (1998) 1642.
[14] S. Taherion, Y.C. Lai, Phys. Rev. E 59 (1999) R6247.
[15] H.U. Voss, Phys. Rev. E 61 (2000) 5115.
[16] T.L. Carroll, J.F. Heagy, L.M. Pecora, Phys. Rev. E 54 (1996) 4676;
L. Kocarev, U. Parlitz, Phys. Rev. Lett. 76 (1996) 1816.
[17] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization—A Universal Concept in Nonlinear Sciences, Cambridge Univ. Press, Cambridge, 2001.
[18] R. Femat, G. Solís-Perales, Phys. Rev. E 65 (2002) 036226.
[19] S. Bowong, Phys. Lett. A 326 (2004) 102.
[20] C. Wang, S.S. Ge, Int. J. Bifur. Chaos 11 (2001) 1743.
[21] Z.M. Ge, W.Y. Leu, Chaos Solitons Fractals 20 (2004) 503.
[22] M. Bazhenov, R. Huerta, M.I. Rabinovich, T. Sejnowski, Physica D 116 (1998) 392.
[23] R. Huerta, M. Bazhenov, M.I. Rabinovich, Europhys. Lett. 43 (1998) 719.
[24] D. Terman, N. Koppel, A. Bose, Physica D 117 (1998) 241.
[25] C. Schäfer, M. Rosenblum, H.H. Abel, J. Kurths, Phys. Rev. E 60 (1999) 857.
[26] K. Kotani, K. Takamasu, Y. Ashkenazy, H.E. Stanley, Y. Yamamoto, Phys. Rev. E 65 (2002) 051923.
[27] U. Parlitz, Phys. Rev. Lett. 76 (1996) 1232.
[28] A. d'Anjou, C. Sarasola, F.J. Torrealdea, R. Orduna, M. Graña, Phys. Rev. E 63 (2001) 046213.
[29] S.H. Chen, J.H. Lü, Phys. Lett. A 299 (2002) 353.
[30] S.H. Chen, J. Hu, C.P. Wang, J.H. Lü, Phys. Lett. A 321 (2004) 50.
[31] M.T. Yassen, Phys. Lett. A 337 (2005) 335.
[32] M.E. Elmetwally, N.A. Hamdy, Int. J. Non-Linear Sci. Numer. Simul. 5 (2) (2004) 171.
[33] J.H. Park, Int. J. Non-Linear Sci. Numer. Simul. 6 (2) (2005) 201.
[34] L. Stenflo, Phys. Scr. 53 (1996) 83.
[35] O.E. Rössler, Phys. Lett. A 7 (1979) 155.
[36] L. Li, H. Peng, X. Wang, Y. Yang, Phys. Lett. A 333 (2004) 269.